

Math 450 - Homework 11

Solutions

1. We fix a time T and wish to observe the distribution of positions of W_T , the one-dimensional Brownian motion. Do this as follows. Using the program `bpath2.m` in Higham's paper, simulate a large number of sample paths of W_t for $0 \leq t \leq T$ and plot the empirical distribution of values of W_T . (Suggestion: simulate 10000 sample paths and calculate the fraction of outcomes over 50 equally spaced bins. This is easily done using the `hist` command. On the same coordinate system plot the graph of the theoretical values that would be obtained under the exact normal distribution with mean 0 and variance T . (Suggestion: use a style of plot similar to that of figure 4 on page 15 of lecture notes set 3.) Do this in two separate graphs, one for $T = 0.1$ and another for $T = 1.0$.

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Distribution of endpoints of Brownian paths
T=1;
N=500;
dt=T/N;
K=10000;
a=[];
s=0.1;
for i=1:K
    dW=sqrt(dt)*randn(1,N);
    W=cumsum(dW);
    a=[a W(ceil(s*N))];
end
[n,xout]=hist(a,50);
hold off
stem(xout,n/K)
dx=xout(2)-xout(1);
fdx=(1/sqrt(2*pi*s))*exp(-xout.^2/(2*s))*dx;
hold on
plot(xout,fdx)
grid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

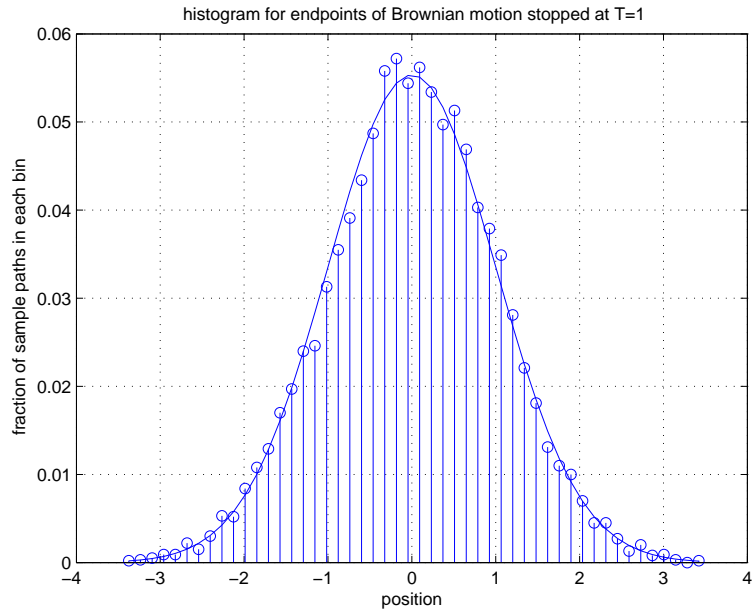
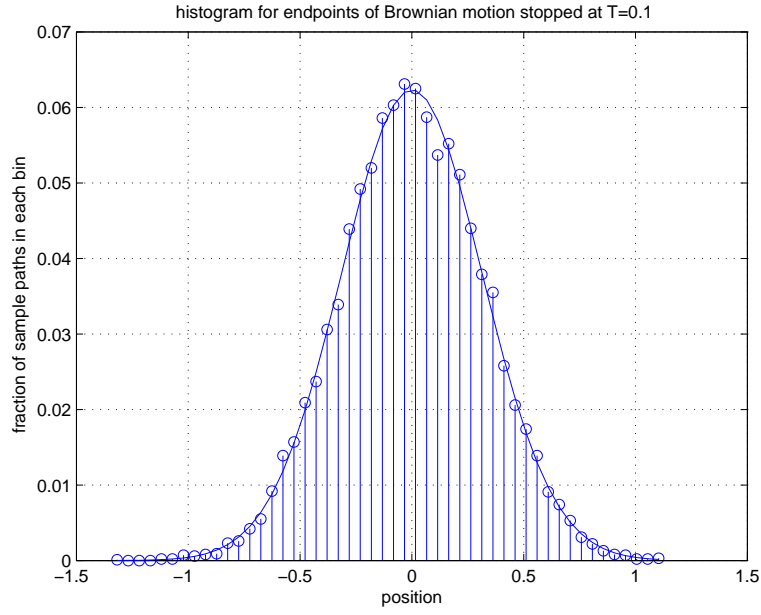


Figure 1: Distribution of X_T , for $T = 0.1$ and $T = 1$, where $t \mapsto X_t$ is one-dimensional Brownian motion.

- Adapt the program `bpath2.m` to obtain sample paths of a (discretized) two-dimensional Brownian path. (Suggestion: use time interval $[0, 1]$ and $N = 10000$ time steps. For a better looking graph, use the `axis equal` command so that the x - and y -coordinate axis display the same length scale.)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%2D Brownian paths
T=1;
N=10000;
dt=T/N;
dW1=sqrt(dt)*randn(1,N);
dW2=sqrt(dt)*randn(1,N);
W1=cumsum(dW1);
W2=cumsum(dW2);
plot(W1,W2)
axis equal
grid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

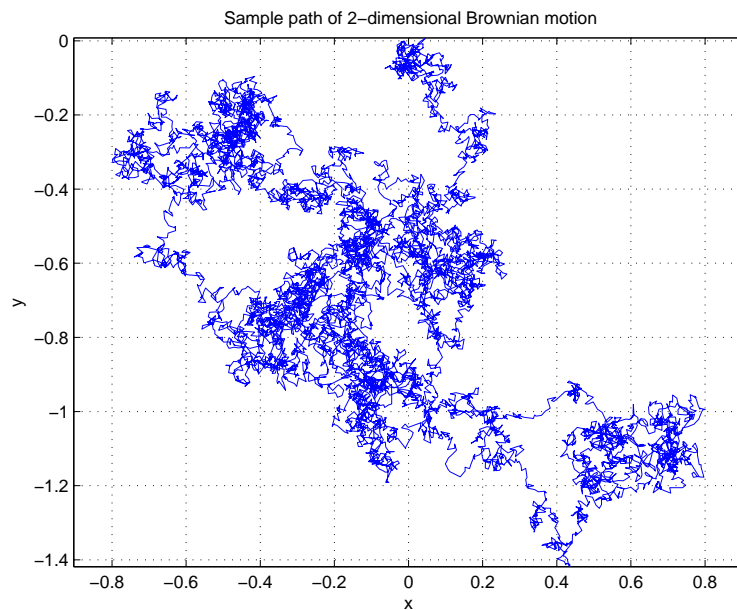


Figure 2: Sample path of two-dimensional Brownian motion.

3. Obtain a plot similar to that of figure 2, page 529 of Higham's paper, but now for the process

$$X_t = \cos(t + W_t).$$

(The graph in that paper corresponds to the process $X_t = \exp(t + \frac{1}{2}W_t)$.)

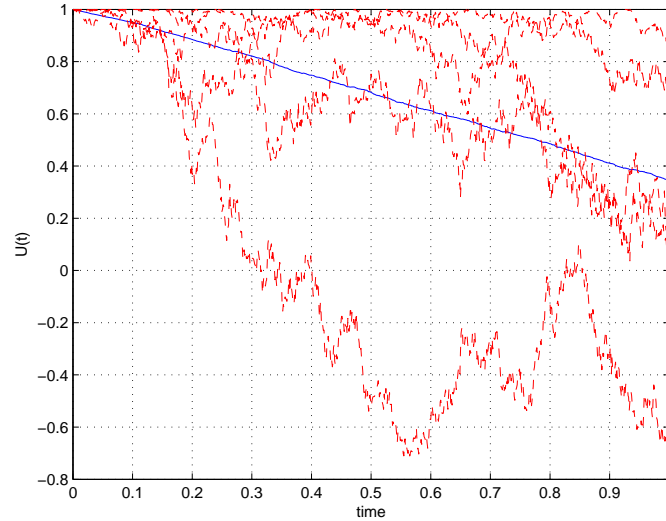


Figure 3: Five sample paths for the process $\cos(t + W_t)$, $0 \leq t \leq 1$, and the average value over 1000 sample paths.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
T=1;
N=1000;
dt=T/N;
t=dt:dt:T;
M=1000; %number of sample paths involved in average
dW=sqrt(dt)*randn(M,N);
W=cumsum(dW,2);
U=cos(repmat(t,[M 1]) + W);
Umean=mean(U);
plot([0,t], [1,Umean], 'b-')
hold on
plot([0,t], [ones(5,1),U(1:5,:)], 'r--')
hold off
grid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

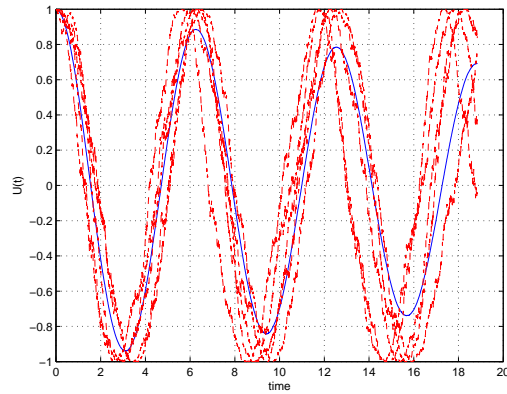


Figure 4: Five sample paths for the process $\cos(t + 0.2W_t)$, $0 \leq t \leq 6\pi$, and the average value over 5000 sample paths.

The next graph only shows the average. Can you explain why the amplitude seems to be decreasing? Would you expect it to decrease all the way to 0?

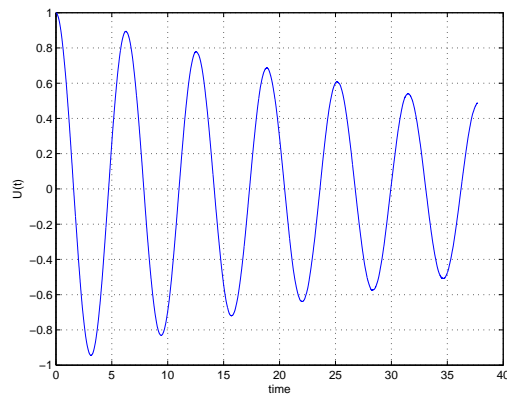


Figure 5: Average over 10000 sample paths of $\cos(t + 0.2W_t)$, $0 \leq t \leq 12\pi$.