Math 450 - Homework 12

Due date: Friday, 4/20/07

1. Read sections 3 and 4 (pages 530-533) of D.J. Higham's paper. Also read the lecture notes posted on-line about stochastic differential equations. (I should have them ready sometime over the weekend.)

2. Use Itô's formula to write the following stochastic processes X_t in standard form,

$$dX_t = u(t)dt + v(t)dW_t.$$

(We denote by $W_t = (W_1(t), \ldots, W_n(t))$ the n-dimensional Wiener process. Part (c) uses the 2-dimensional process, and part (d) uses the 3-dimensional process.)

   (a) $X_t = W_t^3$
   (b) $X_t = 2t + e^{W_t}$
   (c) $X_t = W_1^2(t) + W_2^2(t)$
   (d) $X_t = (W_1(t) + W_2(t) + W_3(t), W_2^2(t) - W_1(t)W_3(t))$

3. Express the Itô integral

$$\int_0^t \sin(W(s))dW(s)$$

as a sum of a function of $W(t)$ and a deterministic integral; i.e., find functions $F(x)$ and $g(x)$ such that

$$\int_0^t \sin(W(t))dW(t) = F(W(t)) + \int_0^t g(W(s))ds.$$

Explain how you did it.

4. Calculate numerically a sample path of the Itô integral

$$A(t) = \int_0^t \sin(W(s))dW(s)$$

for $0 \leq t \leq 1$. (Adapt the program `stint.m` from Higham's paper. Use `N=10000`.) Then plot the sample path for the difference $A(t) - B(t)$, where

$$B(t) = 1 - \cos(W(t)) - \frac{1}{2}\int_0^t \cos(W(s))ds.$$

(Note: it is important to use the same sample path for the Wiener process when calculating $A(t)$ and $B(t)$. We should expect this difference to be small. Why?)

5. In this exercise you will obtain numerical solutions for a stochastic harmonic oscillator. First recall that the deterministic case. We assume a spring-mass system with mass m, spring constant k and friction coefficient c.

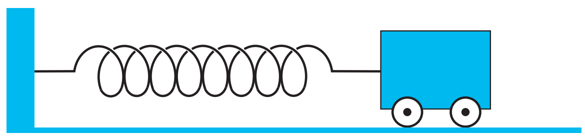

Figure 1: Spring-mass system. We assume that the mass attached to the spring is acted on by a random force term given by white noise.

Recall that the differential equation for this system (obtained from Newton's second law) has the form:

$$mx'' = -kx - cx' + f(t)$$

where $f(t)$ is a forcing term. Let $\kappa = k/m$, $\gamma = c/m$, and $\eta(t) = f(t)/m$. Let $v = x'$. The second order differential equation can be written as a system of two first order equations

$$\begin{aligned} x' &= v \\ v' &= -\kappa x - \gamma v + \eta(t). \end{aligned}$$

We assume that the forcing term is random, of the form $\eta(t)dt = \sigma dW_t$. This leads to the system of stochastic differential equations written in matrix form as:

$$\begin{pmatrix} dX_t \\ dV_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa & -\gamma \end{pmatrix} \begin{pmatrix} X_t \\ V_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t.$$

Take $\kappa = 1$, $\gamma = 0.1$ and $\sigma = 0.1$. Adapt the Euler-Maruyama method (see `em.m` in Higham's paper) to solve for a sample path X_t, $0 \le t \le T$, $T = 100$, with initial conditions: $X_0 = -1$, $V_0 = 0$. (Choose $dt = T/10000$ for the step size.) On the same coordinate system plot the graph of X_t versus t for 10 independent sample paths.