Practice Differentiations Using the Fundamental Theorem of Calculus, Part I

What is $\frac{d}{ds}\int_1^2 s^2\, ds$?

Solution: $\int_1^2 s^2\, ds$ is a constant $(=\frac{7}{3}$, if you work it out$)$, and $\frac{d}{ds}$(constant) $= 0$.

What is $\frac{d}{ds}\int_1^s t^2\, dt$?

Solution: By the Fundamental Theorem of Calculus (Part I),

$\frac{d}{ds}\int_1^s t^2\, dt\ = s^2$ (In this case, the integral is so easy that you could simplify it first and avoid the Fundamental Theorem: $\frac{d}{ds}\int_1^s t^2\, dt\ = \frac{d}{ds}(\frac{s^3}{3} - \frac{1}{3}) = s^2$.)

What is $\frac{d}{dz}\int_1^s t^2\, dt$?

Solution: $\int_1^s t^2\, dt$ is a function of s, not of z: so $\frac{d}{dz}\int_1^s t^2\, dt = 0$ (*)

Actually, this assumes that s and z are independent $-$ in other words, that s does not depend on z so that $\int_1^s t^2\, dt$ also does not depend on z and therefore $\frac{d}{dz}\int_1^s t^2\, dt = 0$.

If you want to allow for the possibility that s might depend on z (for example, that perhaps $s = z^2$), we would use the chain rule:

$$\frac{d}{dz}\int_1^s t^2\, dt = s^2 \cdot \frac{ds}{dz}. \quad (**)$$

This formula (**) "better than" our original solution (*) because, if s is really independent of t, then $\frac{ds}{dt} = 0$ and (**) reduces to the original formula (*)

$$\frac{d}{dz}\int_1^s t^2\, dt = 0.$$

What is $\frac{d}{du}\int_u^{2u} 2t\, dt$?

Solution: $\frac{d}{du}\int_u^{2u} 2t\, dt = \frac{d}{du}\int_u^{0} 2t\, dt + \frac{d}{du}\int_0^{2u} 2t\, dt$
$= \frac{d}{du}\int_0^{2u} 2t\, dt - \frac{d}{du}\int_0^{u} 2t\, dt = \frac{d}{du}\int_0^{2u} 2t\, dt - 2u$
$= 2(2u)\frac{d}{du}(2u) - 2u = 8u - 2u = 6u.$

What is $\frac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt$?

Solution: Since both limits on the integral are the same, $\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1} = 0,$ so $\frac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt \; = \frac{d}{dz}(0) = 0.$

(Of course, you could work it out the "longer way":

$$\begin{aligned} &\tfrac{d}{dz}\int_{z^3}^{z^3} \sqrt[4]{t^2+t+1}\,dt \\ &\quad = \tfrac{d}{dz}\int_{z^3}^{0} \sqrt[4]{t^2+t+1}\,dt + \tfrac{d}{dz}\int_{0}^{z^3} \sqrt[4]{t^2+t+1}\,dt \\ &\quad = -\tfrac{d}{dz}\int_{0}^{z^3} \sqrt[4]{t^2+t+1}\,dt + \tfrac{d}{dz}\int_{0}^{z^3} \sqrt[4]{t^2+t+1}\,dt = 0.) \end{aligned}$$

What is $\frac{d}{ds}\int_1^{\int_1^s t^2\,dt} t\,dt$?

Solution: Let $y = \int_1^{\int_1^s t^2\,dt} t\,dt = \int_1^u t\,dt,$ where $u = \int_1^s t^2\,dt.$

Then $\frac{du}{ds} = s^2$ (using the Fundamental Theorem, Part I)) so

$$\frac{dy}{ds} = \frac{dy}{du}\frac{du}{ds} = u \cdot \frac{du}{ds} = u \cdot s^2 = s^2 \int_1^s t^2\,dt$$

If we simplify further, $s^2\int_1^s t^2\,dt = s^2 \cdot \frac{t^3}{3}\Big|_1^s$

$$= s^2\left(\frac{s^3}{3} - \frac{1}{3}\right) = \frac{s^5 - s^2}{3}.$$