Q1: Let $y = f(x) = \sqrt{x}$. Suppose we use differentials to approximate both $\sqrt{4.1}$ and $\sqrt{100.1}$ (using $dx = 0.1$ in both calculations). Which of the following is true?
(*"magnitude of the approximation error" means |approximate value − exact value|*)

A) In both cases, the approximation is smaller than the actual value, but the magnitude of the approximation error is larger in the case of $\sqrt{4.1}$

B) In both cases, the approximation is smaller than the actual value, but the magnitude of the approximation error is larger in the case of $\sqrt{100.1}$

C) In both cases, the approximation is larger than the actual value, but the magnitude of the approximation error is larger in the case of $\sqrt{4.1}$

D) In both cases, the approximation is larger than the actual value, but the magnitude of the approximation error is larger in the case of $\sqrt{100.1}$

Answer: The tangent line is always above the graph of $y = \sqrt{x}$, so the linear approximation (same as the approximation using differentials, in different notation) is always larger than the actual value.

When x is larger, the tangent line to $y = \sqrt{x}$ is closer and closer to being horizontal and the tangent line "hugs" the graph of $y = \sqrt{x}$ more closely, so the error is smaller at $x = 100$, or, put the other way, the error is larger when approximating $\sqrt{4.1}$.

Example: Use the method of differentials to estimate the value of $\sqrt{5248}$.

Let $y = f(x) = \sqrt{x}$, so that $dy = \frac{1}{2\sqrt{x}}dx$

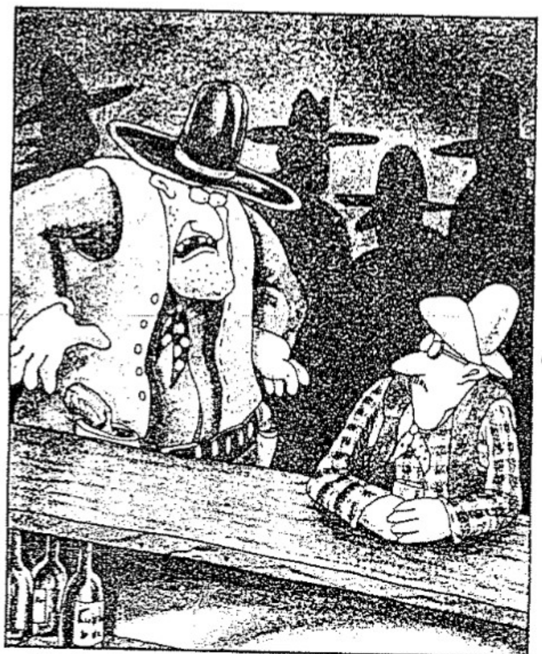
Use $x = 4900$ and $dx = 348$ (*<u>not</u> a small* dx)

Then $\sqrt{5248} = f(x + dx)$

$$\begin{aligned} &= f(x) + \Delta y \\ &\approx f(x) + dy \\ &\approx f(4900) + dy \\ &= 70 + \tfrac{1}{2\sqrt{4900}}(348) \approx 70 + \tfrac{1}{2(70)}(350) \\ &= 72.5 \end{aligned}$$

For comparison, a calculator gives $\sqrt{5248} \approx 72.44$!

Even with such a large dx, we got a reasonable approximation <u>in this case</u> because for x's as large as 4900, the function $y = \sqrt{x}$ is very close to running horizontally, and the error in a linear approximation, even with such a "large" $dx = 348$ is actually very small. This is the same phenomenon as made the error in approximating $\sqrt{100.1}$ smaller than the - error for $\sqrt{4.1}$ – except even more dramatic!



"I asked you a question, buddy. . . . What's the square root of 5,248?"

$$y=\sqrt{x},\quad dy=\frac{1}{2\sqrt{x}}\,dx$$

$$f(4900+348)-f(4900)\simeq dy$$

$$\sqrt{5248}\simeq 70+\frac{1}{2(70)}(348)$$

$$\simeq 70+\frac{1}{2(70)}(350)$$

$$=70+\frac{5}{2}=72.5$$

Differentials save the day!

Q2: (*Answer without calculating: the point of this question is just to check your intuition! Assume the earth is a sphere, with radius 4000 miles.*)

You wrap a string around the earth at the equator – so the length of the string will equal the circumference of the earth.

Then you decide that, instead, you'd like to have the string always 6 inches above the ground (*on tiny utility poles planted along the equator, perhaps?*) How much longer will the string need to be?

A) About 3 inches

B) About 3 feet

C) About 3 miles

D) About 30 miles

E) About 300 miles

Answer: Express units in feet. But there's no need to actually convert $r = 4000$ *miles into feet since it doesn't actually appear in the calculation:*

$$C = 2\pi r \qquad \text{so } \frac{dC}{dr} = 2\pi \qquad \text{so } dC = 2\pi dr$$

When the radius of the circle is increased by $\frac{1}{2}$ *ft, then the change in the circumference is*

$$\Delta C \approx dC = 2\pi(\tfrac{1}{2}) = \pi \text{ ft.} \approx 3.14 \text{ ft.}$$

(The slightly hard calculation would convert 4000 miles into $21,120,000$ *ft and then write*

$$\Delta C = C(21120000.5) - C(21120000) = \ldots$$

(In this particular example, it turns out that $\Delta C = dC$*: why does that happen here?)*

$y = f(x)$ with domain D

D might be the "natural" domain $=$ the set of all x's for which the formula makes sense. For example, $f(x) = \frac{1}{\sqrt{x^2-4}}$ has "natural" domain $D = (-\infty, 2) \cup (2, \infty)$
The formula doesn't make sense for x's in the interval $[-2, 2]$

But for some reason (perhaps constraints in a physical application) someone might say: consider the function $f(x) = \frac{1}{\sqrt{x^2-4}}$ using the domain $D = [3, 8]$. In this case, the domain is smaller than the "natural" domain.

Suppose c is in the domain D of $f(x)$

$f(c)$ is called the <u>absolute maximum value</u> for $f(x)$ on D if $f(c) \geq f(x)$ for all x in D

$f(c)$ is called the <u>absolute minimum value</u> for $f(x)$ on D if $f(c) \leq f(x)$ for all x in D

$f(c)$ is called a <u>local maximum value</u> for $f(x)$ if $f(c) \geq f(x)$ <u>when</u> x is <u>near</u> c (on both sides of x)

$f(c)$ is called a <u>local minimum value</u> for $f(x)$ if $f(c) \leq f(x)$ <u>when</u> x is <u>near</u> c (on both sides of x)

> *Note: "local maximum value" and "local minimum value" require that some inequality be true <u>near x on both sides of x</u>. So if the domain D has one or more <u>endpoints</u>, a local maximum value or local minimum value cannot occur at an endpoint of D.*

If we don't want to distinguish between maxima and minima, we might "lump them together" and take about

absolute <u>extreme values</u> to refer to both absolute maximum value and absolute minimum value

local <u>extreme values</u> to refer to both local maxima and local minima

(*These definitions were illustrated in class, as they are in the textbook.*)

The illustrations led us to conjecture when there is a local maximum or minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.