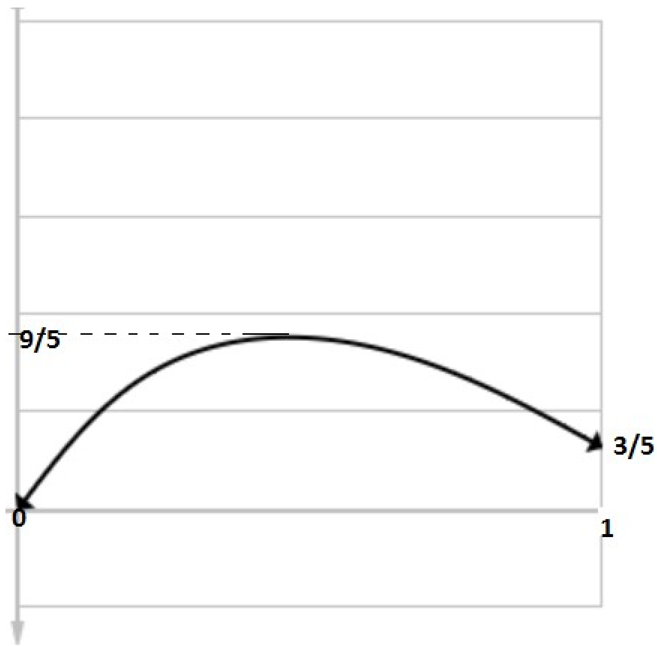
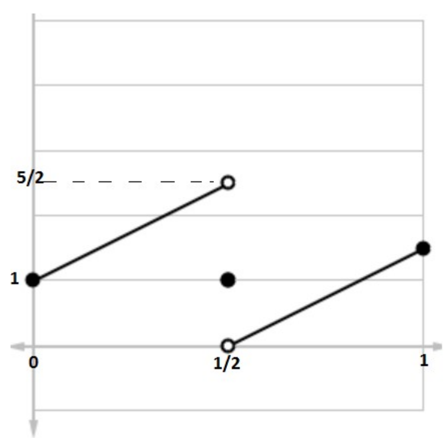
Exercise: Draw the graph of a function $y = f(x)$ whose domain is $[0, 1]$, that has an absolute maximum and minimum value. In your picture, the absolute minimum should occur at an endpoint, and the absolute maximum value not at an endpoint.



The absolute minimum value is 0, and it occurs at 0
The absolute maximum value is $\frac{9}{5}$ occurring at $\frac{1}{2}$

Exercise: Draw the graph of a function $y = f(x)$ whose domain is $[0, 1]$, that has no absolute maximum value and no absolute minimum value.



The function has no absolute maximum value: $\frac{5}{2}$ is not the absolute maximum value because $\frac{5}{2}$ is not even a value: there is no c in $[0, 1]$ where $f(c) = \frac{5}{2}$! Similarly, there is no absolute minimum value.

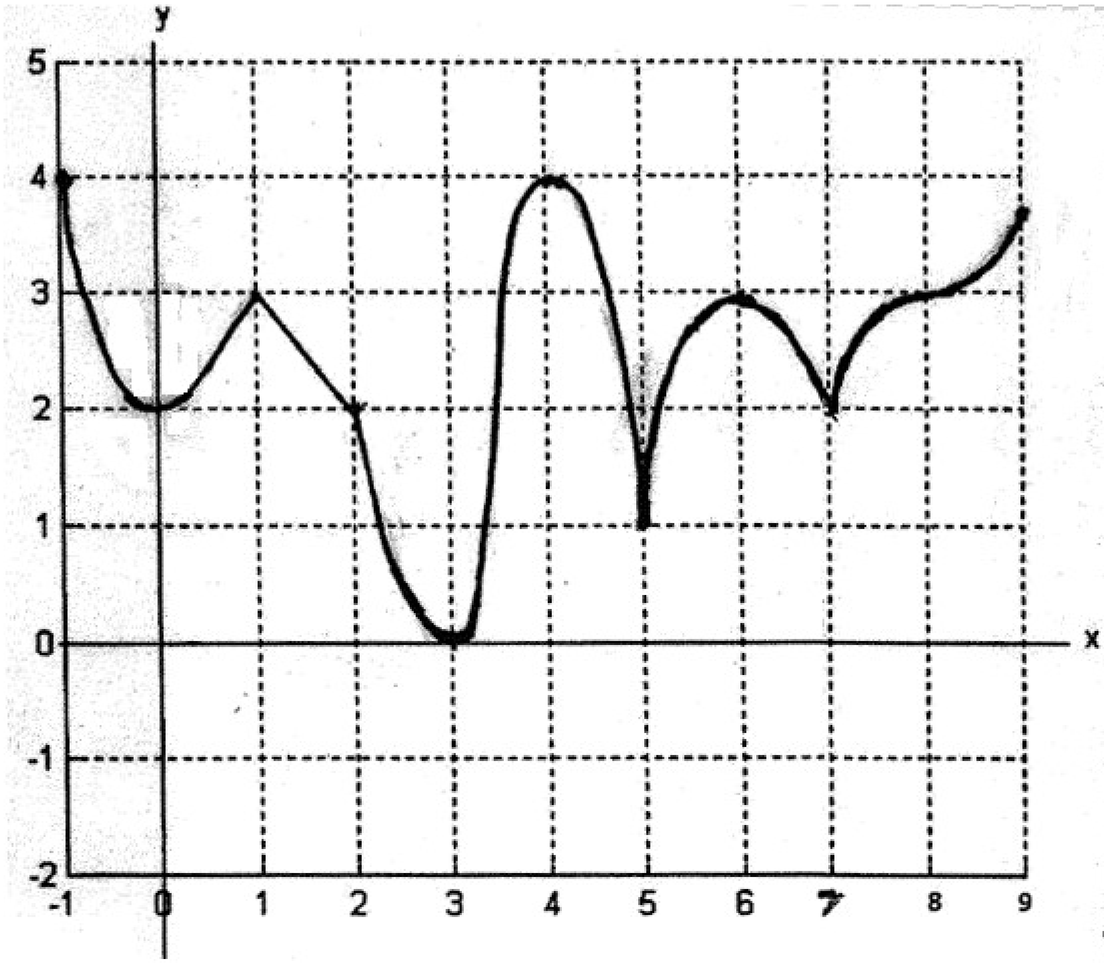
If you tried to draw a continuous f to answer this exercise, that's IMPOSSIBLE!

It's impossible because of a (rather deep, hard to prove) theorem:

The Extreme Values Theorem

If f is a continuous function whose domain is a closed interval $[a, b]$, then

f must have both an absolute maximum value and an absolute minimum value.



there is an absolute maximum value at what x's ? $\boxed{-1,\ 4}$

there is an absolute minimum value at what x's ? $\boxed{3}$

there is a local maximum value at what x's ? $\boxed{1,\ 4,\ 6}$

there is a local minimum value at what x's ? $\boxed{0,\ 3,\ 5,\ 7}$

$f'(x) = 0$ at what x's ? $\boxed{\mathbf{0,\ 3,\ 4,\ 6,\ 8}}$

$f'(x)$ does not exist at what x's ? $\boxed{\mathbf{1,\ 2,\ 5,\ 7}}$

The x's (*in the domain of f*) where either $f'(x) = 0$ or $f'(x)$ does not exist are called the **critical numbers** for the function f: $\boxed{\mathbf{0,\ 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8}}$

Notice that

i) all the x's for which there is a local maximum $(1,\ 4,\ 6)$ or a local minimum $(0,\ 3,\ 5,\ 7)$ appear in the list of critical values, and

ii) that there are some critical numbers $(2,\ 8)$ which do not correspond to either a local maximum or minimum. They are "losers."

iii) an absolute maximum or minimum is also a local maximum or minimum UNLESS it happens to occur at an endpoint – because the convention in our textbook is that local maxima/minima can't happen at endpoints of the domain (*if there are any endpoints to the domain*).

The preceding picture and the chart illustrate the content of Fermat's Theorem (which can be stated in two different (but equivalent ways):

(*) **Fermat's Theorem**: **If** f has a local maximum or minimum at c **then** $f'(c) = 0$
and $f'(c)$ exists

Logically, this can be reworded to say the same thing as follows:

(**) If f has a local max value or local min value at $x = c$, then either $\begin{cases} f'(c) = 0, & \text{or} \\ f'(c) \text{ does not exist} \end{cases}$

This gives us an easy procedure to starting hunting for local maxima and minima:

For a function $y = f(x)$ with domain D :

1) to get a list of the **candidates** $\boldsymbol{c}$ at which where f **might** have a local maximum or minimum:

Find all c's in the domain (not endpoints of D) for which $f'(c) = 0$ or $f'(c)$ doesn't exist. The numbers in this list are called the critical numbers or critical points for f)

2) to get list of **candidates** $\boldsymbol{c}$ at which f **might** have an absolute maximum or absolute minimum value:

List all the critical numbers of f and, in addition, the endpoints (*if any*) of the domain D.

Example: Where <u>might</u> $f(x) = f(x) = \frac{x-1}{x^2-x+1}$ have a local maximum or minimum?
Answer: The list of candidates = the list of all the critical numbers for $f(x)$

Notice that the denominator is never 0 : setting $x^2 - x + 1 = 0$ and applying the quadratic formula shows that there are no (real) solutions for x. Therefore the natural domain of the function is $D = (-\infty, \infty)$, that is, $D =$ the set of all real numbers.

$$f'(x) = \frac{(x^2-x+1)(1)-(x-1)(2x-1)}{(x^2-x+1)^2} = \frac{x^2-x+1-2x^2+3x-1}{(x^2-x+1)^2} = \frac{-x^2+2x}{(x^2-x+1)^2}$$

$$= \frac{-x(x-2)}{(x^2-x+1)^2} = 0. \quad \text{Therefore}$$

i) $f'(x) = 0$ has solutions 0 and 2: so 0 and 2 are critical numbers.

ii) Since the denominator is never 0, the derivative exists at every x. Therefore there are no additional critical numbers.

Therefore f <u>might</u> have a local maximum or minimum at 0 and at 2. But there are <u>no other possible</u> candidates.

Question: Which candidates are "winners" – which critical numbers (*if any*) really do produces a local maximum or a local minimum for f?

We will look in more depth later about how to decide whether each critical number <u>actually</u> corresponds to a local maximum of minimum. In this simple example, however, look at the <u>sign of the derivative</u>.

$$f'(x) = \frac{-x(x-2)}{(x^2-x+1)^2}, \text{ so } \frac{\begin{cases} \text{numerator negative} & \text{if } x > 2 \\ \text{numerator positive} & \text{if } 0 < x < 2 \\ \text{numerator negative} & \text{if } x < 0 \end{cases}}{\text{denominator always positive}}$$

so on $(-\infty, 0)$: $f'(x)$ is negative, so $f(x)$ has tangent lines with negative slopes, so f is decreasing

on $(0, 2)$: $f'(x)$ is positive. so $f(x)$ has tangent lines with positive slopes, so f is increasing

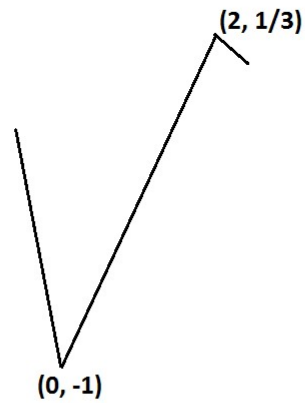
on $(2, \infty)$: $f'(x)$ is negative, so $f(x)$ has tangent lines with negative slopes, so f is decreasing.

Compute $f(0) = -1$, $f(2) = \frac{1}{3}$.

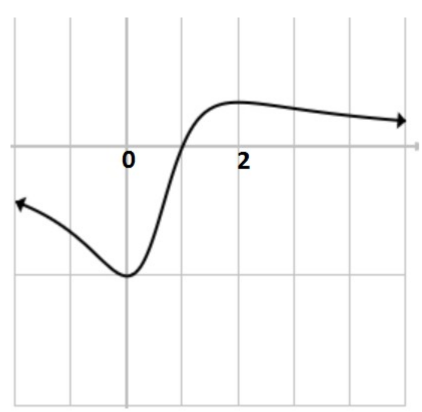
Putting all this together (*without drawing a "careful" graph*) we can say that as

> to the left of $0:\ f(x)$ <u>decreases</u> toward $f(0) = -1$, and just to the right of 0, f starts to <u>increase</u>: this "shows" me that there must be a <u>local minimum</u> at $x = 0$.
>
> just to the left of 2, f <u>increase</u> toward $f(2) = \frac{1}{3}$, and just to the right of 2, f starts to <u>decrease</u> again: this "shows" me that there must be a <u>local maximum</u> at $x = 2$.

A "schematic diagram"of the function's up-down behavior looks like:



A precise picture (just for reference) is



Example Where might $f(x) = \frac{x-1}{x^2-x+1}$ with domain $D = [\frac{1}{2}, 3]$ have an absolute maximum or minimum value?

Answer: Since f is a continuous function on a closed interval D, the Extreme Value Theorem guarantees the it has an absolute maximum and minimum value.

An absolute maximum/minimum might occur where there is a local maximum/minimum, OR at an endpoint of the domain.

So the list of candidates for where an absolute maximum or minimum might occur is

the list of all the critical critical numbers in D $+$ the endpoints of D.

Using the same derivative formula as in the preceding example, but restricting our attention to the domain D:

$f'(x) = 0$ the only solution in D is 2, and there are no x's in D for which the derivative does not exist.

So the list of candidates for where to find an absolute maximum or minimum is:

2 (the only critical point), and $\frac{1}{2}$, 3 (the endpoints of the domain D)

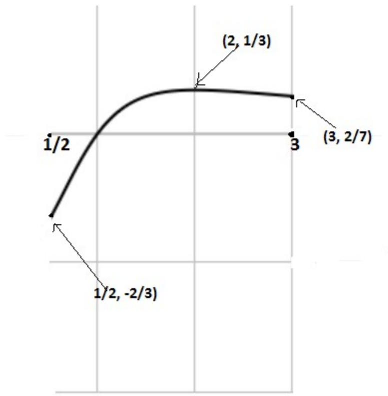
In this case, the Extreme Value Theorem guarantees in advance that one in the list must be the location for the absolute max value, and one for the absolute min value. We can find the "winners" just be checking the value of f at each candidate:

$$
\begin{aligned}
f(\tfrac{1}{2}) &= -\tfrac{2}{3} \\
f(2) &= \tfrac{1}{3} \\
f(3) &= \tfrac{2}{7} \quad (\approx 0.2857)
\end{aligned}
$$

From the list the largest value is $\frac{1}{3}$, occurring at $x = 2$: $\frac{1}{3}$ must be the absolute maximum value

From the list, the smallest value is $-\frac{2}{3}$, occurring at $x = \frac{1}{2}$: $-\frac{2}{3}$ must be the absolute minimum value.

You can see these on the graph of f (where the domain is restricted now to $D = [\frac{1}{2}, 3]$).



(2, 1/3)
1/2
3
(3, 2/7)
1/2, -2/3)