Review problem:

Find the absolute maximum and minimum values of the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-1, 2]$.

We know that there is an absolute maximum value and there is an absolute minimum value because f is a continuous function and its domain is a closed interval.

$f'(x) = \frac{-2x}{1+x^2}$. The derivative is defined for all x, and $f'(x) = 0$ only when $x = 0$.

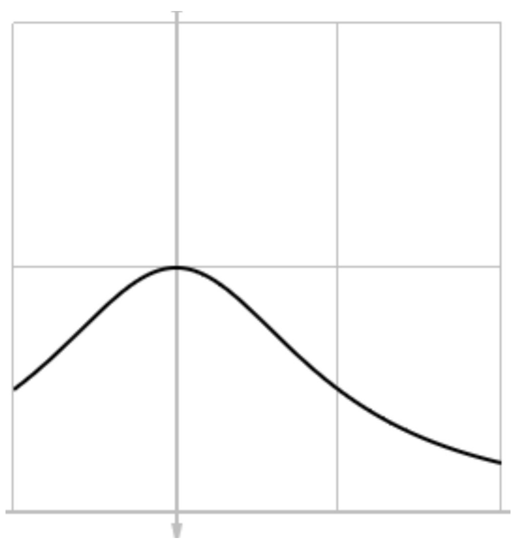
The candidates for where the absolute maximum and minimum of f might occur:

Critical numbers: 0
Endpoints: $-1, 2$

The absolute maximum and minimum values must occur at one of these points.

Since $f(-1) = \frac{1}{2}$, $f(0) = 1$, and $f(2) = \frac{1}{5}$, we conclude that the absolute max value is 1 (occurring at $x = 0$) and the absolute minimum value is $\frac{1}{5}$ (occurring at $x = 2$).

Notice that $f'(x) > 0$ *when* $x < 0$ *and that* $f'(x) < 0$ *when* $x > 0$.
This means that f *is increasing when* $x < 0$ *and decreasing when* $x > 0$.
Just for reference, the graph of $y = f(x)$ *looks like*

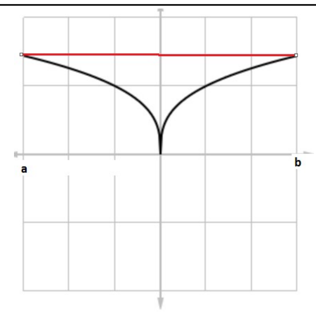
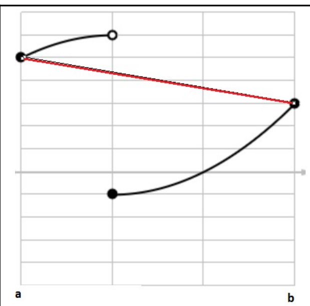
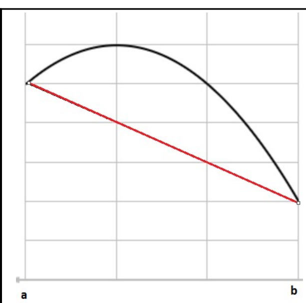
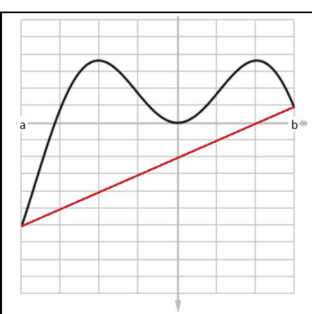


Each picture shows the graph of a function $y = f(x)$ and (in red) the straight line segment L joining $(a, f(a))$ to $(b, f(b))$.

In each case: is there a point $(z, f(z))$ on the graph of $y = f(x)$ where **slope of tangent line to graph of f** $=$ **slope of line segment L** ?

or equivalently

is there a z in (a, b) where $\quad f'(z) = \dfrac{f(b) - f(a)}{x - a}$?

Q1		A) yes **B) no**
Q2		A) yes **B) no**
Q3		**A) yes** B) no
Q4		**A) yes** B) no

In Q1 and Q2, the function is either not differentiable; in Q2, the function is not even continuous. In those two pictures, no such z exists; in the picture for Q3 and Q4, the function is continuous and differentiable (in the "interior" of the domain – that is, between then endpoints. In Q3, there is a z that works; in Q4 there is more than one z that works.

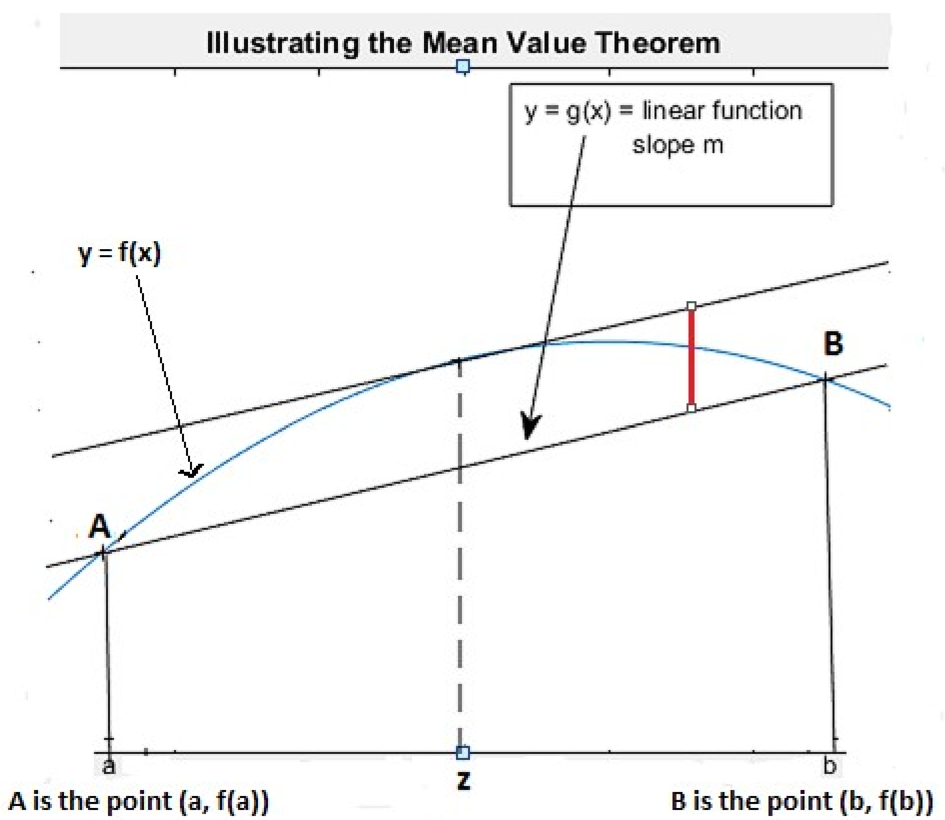
The Mean Value Theorem tells us when such a z (maybe more than one) is guaranteed to exist:

The Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable for all x between a and b, then
there must be <u>at least one</u> z in (a, b) where

$$f'(z) = \frac{f(b) - f(a)}{b - a} \text{ or, equivalently, where}$$

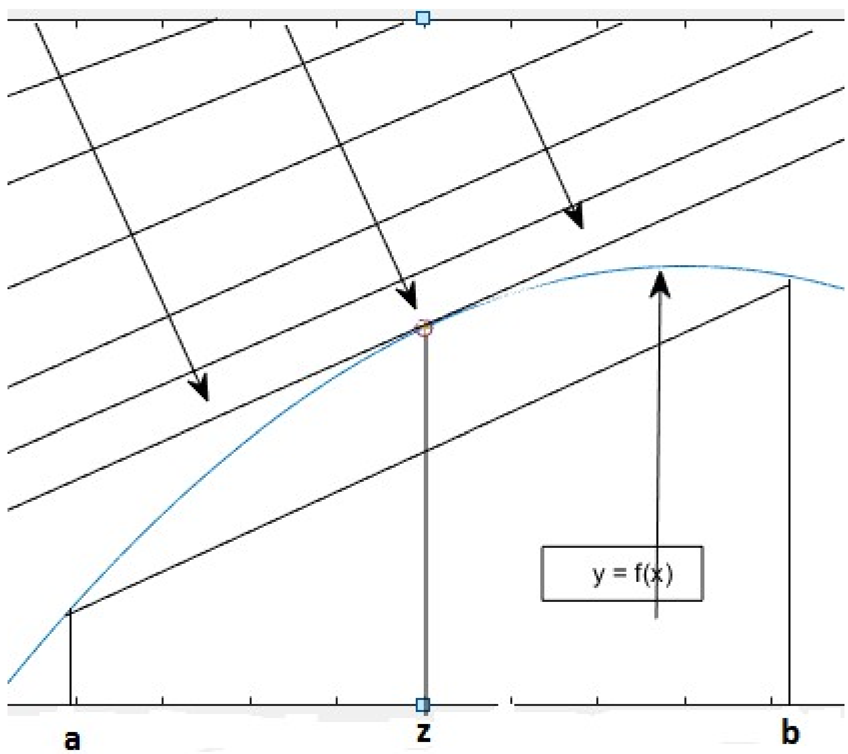
$$f'(z)(b - a) = (f(b) - f(a))$$



Geometric process indicating why the Mean Value Theorem is plausible

Move away from the graph of $y = f(x)$ and draw a straight line parallel to the secant line through the points $A = (a, f(a))$ and $B = (b, f(b))$.

The slide your line toward the graph (following the arrows), keeping your line parallel to the line through A and B. At the point where your line first touches the graph of f, your line will be a tangent line parallel to the line through A and B. The first coordinate of this contact point is a z that works.



$f'(z) = (f(b) - f(a)) /(b-a)$

If we change the name of the variables: suppose $s = f(t)$ in the Mean Value Theorem, where s is the position function of a point moving along a straight line. Then $\frac{f(b)-f(a)}{b-a}$ represents the average velocity of the moving point between times $t = a$ and $t = b$.

The Mean Value Theorem says, then, that there must be (at least one) time z between a and b when the instantaneous velocity equals the average velocity:

$$v(z) = f'(z) = \frac{f(b)-f(a)}{b-a}$$

Q5 A particle moves along a straight line for $0 \le t \le 5$).
At time t (sec), its position is $s = f(t) = t^3 - 12t + 70$ (meters).

For your convenience: $\frac{f(5)-f(0)}{5-0} = \frac{135}{5} = 27$ (m/sec)

<u>At what time t</u> is the particle's instantaneous velocity equal to its average velocity for the whole trip?

A) $t = 1$ sec

B) $t = \sqrt{2}$ sec

C) $t = \frac{1+\sqrt{70}}{4}$ sec

D) $t = \sqrt{13}$ sec

E) $t = \sqrt{53}$ sec

Answer Instantaneous velocity $= v(t) = \frac{ds}{dt} = 3t^2 - 12 = 27$ when $3t^2 = 39$, that is, when $t = \sqrt{13}$ (since $t \ge 0$)

In the Mean Value Theorem, it could happen that the value of f at the endpoints is the same: $f(a) = f(b)$

This is nothing but a "special case" of the Mean Value Theorem, but it is sometimes given a special name:

Rolle's Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable for all x between a and b, and
if $f(a) = f(b)$

there must be at least one z in (a, b) where

$$f'(z) = 0 \quad \left(= \frac{f(b) - f(a)}{b - a}\right)$$

Example Suppose we are trying to solve some equation

$f(x) = k$ (*where f is a differentiable function*)

Whenever we have two roots (solutions), say $x = a$ and $x = b$,
then (according to Rolle's Theorem), there must be a root of the equation

$f'(x) = 0$ between a and b (*the root is the z promised by Rolle's Theorem*)

In other words: between any two roots of $f(x) = k$, there is a root of $f'(x) = 0$.

To be specific:

The equation $x^7 - 14x + 23 = 10$ has at most 3 real roots (*maybe fewer*)

Reason: Suppose the equation had 4 real roots $r_1 < r_2 < r_3 < r_4$

Then the equation

$f'(x) = 0$
$7x^6 - 14 = 0$ would need to have at least 3 real roots
(*one in each of the intervals* (r_1, r_2), (r_2, r_3), *and* (r_3, r_4))
But $7x^6 - 14 = 0$ has only two real roots: $x = \pm\sqrt[6]{2}$.