The Mean Value Theorem

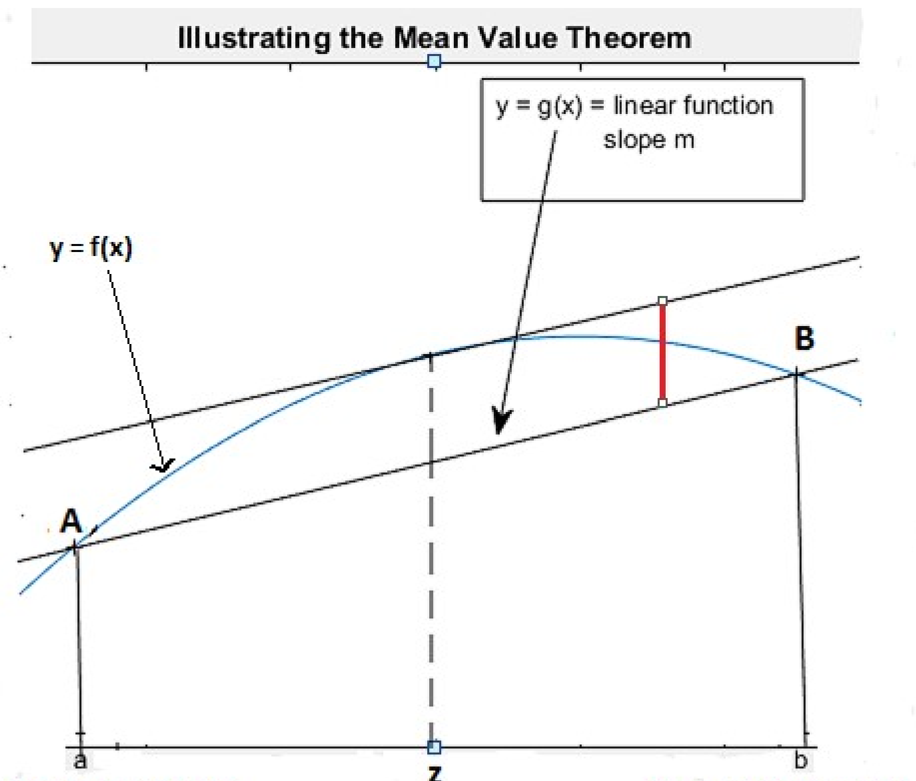
If $f(x)$ is continuous on $[a,b]$ and differentiable for all x in (a,b)
then
there must be at least one z in (a,b) where

instantaneous rate of change at z   *average rate of change of f on* $[a, b]$

$$\underset{f'(z)}{\downarrow} = \underset{\frac{f(b)-f(a)}{b-a}}{\downarrow}$$

or, equivalently, where

$$f'(z)(b-a) = (f(b)-f(a))$$



A is the point (a, f(a))   B is the point (b, f(b))

Example Suppose f is continuous on $[3,7]$ and differentiable on $(3,7)$

$$f(3) = 5 \text{ and}$$

$$-2 \leq f'(x) \leq 4 \text{ at every point } x \text{ in } (3,7)$$

What can you say about the size of $f(7)$?

By the Mean Value Theorem,

$$f(7) - f(3) = f'(z)(7-3) \quad \text{for some } z \text{ in } (3,7)$$

$$f(7) = 5 + 4f'(z) \qquad \text{for some } z \text{ in } (3,7)$$

Replacing the unknown value $f'(z)$ by the smallest and largest possible values, we get

$$f(7) = 5 + 4(-2) \leq f(7) = 5 + 4f'(z) \leq f(7) = 5 + 4(4)$$

so $$-3 \leq f(7) \leq 21$$

If $f(x)$ were the position function at time x $(3 \leq x \leq 7)$ for a point moving along a straight line, then $-2 \leq f'(x) \leq 4$ would mean that the velocity of the moving point was always between -2 and 7. So if this point starts at position 5 (when $x = 3$), then its final position (when $x = 7$) must be somewhere between -3 and 21.

Q1) For $f(x) = x^2 - 3x + 5$ on $[0, 4]$

Find the point (or points) z promised by the Mean Value Theorem

A) $z = 1$

B) $z = 2$

C) $z = \frac{3}{2}$

D) $z = 1$ and 2

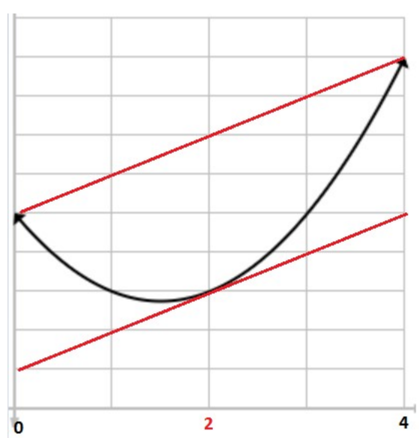
E) $z = \frac{3}{2}$ and $z = \frac{5}{2}$

Answer: The Mean Value Theorem says that

$\frac{f(4) - f(0)}{4 - 0} = f'(z) = 2z - 3$ *for some* z *between* 0 *and* 4.

$\frac{9-5}{4} = 1 - 2z - 3$ *so* $z = 2$

The graph shows the parabola on $[0, 4]$ together with the secant line through the endpoints of the graph and the parallel tangent line at the point $(2, f(2))$.



If you like, you can work out, in the same way, that for any parabola $y = f(x) = ax^2 + bx + c$, considered on the interval $[u, v]$, the point z guaranteed to exist by the Mean Value Theorem is <u>always</u> at the midpoint of the interval: $z = \frac{u+v}{2}$. This is a "geometric" property of parabolas.

Theorem Suppose $f(x)$ and $g(x)$ are differentiable on an interval I.
(*"Interval" means any subset of the real numbers that is "all in one piece"* $-$ *a "connected" subset of the real numbers. Any interval might be finite in length like* $[-2, 6)$ *or infinite in length like* $(-\infty, 3)$ *or even the whole real line* $(-\infty, \infty)$.)

Then

1) if $f'(x) = 0$ for all x in I, then $f(x) = C$ (a constant function) for all x in I

2) if $f'(x) = g'(x)$ for all x in I. then $f(x) = g(x) + C$ for all x in I.

Why is 1) true? Pick any two ponts u, v in I with, say, $u < v$. By the Mean Value Theorem, applied on the interval $[u, v]$

$$\begin{aligned} & f(v) - f(u) = f'(z)(v-u) && \text{for some } z \text{ between } u \text{ and } v \\ & f(v) - f(u) = 0 \cdot (v-u) = 0 && \text{since } f' \text{ is always } 0 \\ \text{so} \quad & f(v) = f(u) \end{aligned}$$

Since f has the same value at any two points u, v in I, then f must be constant on $I/$

Why is 2) true: Let $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x) = 0$ in I.
Since $h'(x) = 0$, we see that $h(x) = C$ on I
So $f(x) - g(x) = C$ and therefore $f(x) = g(x) + C$.

Q2) Suppose $f'(x) = 6x^2$ on $\mathbb{R} = (-\infty, \infty)$
and $f(0) = 17$

What is $f(1)$?

A) -3

B) 5

C) 19

D) 23

E) 12

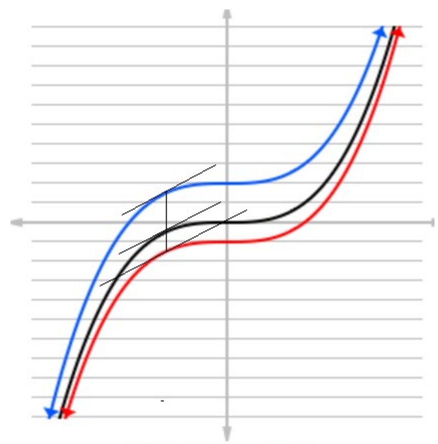
Answer We can guess that $2x^3$ *is a function whose derivative is* $6x^2$.

By our preceding Theorem, all functions with derivative $6x^2$ *must differ from* $2x^3$ *by a constant* :

so $\quad f(x) = 2x^3 + C$

But $\quad 17 = f(0) = 2(0)^3 + C$*, so* C *must be 17*

$f(x) = 2x^3 + 17$*, so* $f(1) = 19$.



Middle Graph: y = 2x^3

Viewing the solution geometrically:

The graph shows the graphs of
$$y = 2x^3 + 4$$
$$y = 2x^3$$
$$y = 2x^3 - 2$$

Geometrically, changing C slides the graph of $y = 2x^3$ up or down by an amount C. C is the y-intercept for $y = f(x) = 2x^3 + C$. Requiring that $f(0) = 17$ picks out one curve from among all the curves $y = f(x) = 2x^3 + C$ for which $f'(x) = 6x^2$: namely, the one that has y-intercept 17.

Definition. For a function f defined on an interval I :

i) we say that f is <u>increasing on I</u> if : whenever $u < v$ in I, then $f(u) < f(v)$
ii) we say that f is <u>decreasing on I</u> if : whenever $u < v$ in I, then $f(u) > f(v)$

Assuming that f is differentiable, we can the use the Mean Value Theorem to see that:

i) If $f'(x) > 0$ on an interval I, then $f(x)$ is <u>increasing on I</u>
ii) If $f'(x) < 0$ on an interval I, then $f(x)$ is <u>decreasing on I</u>

<u>First Derivative Test for Local Maxima and Minima</u>

Suppose $f(x)$ is a continuous function and that c is a critical number:

$$\text{If } \begin{cases} f'(x) > 0 & \text{near } c \text{ on the left} \\ f'(x) < 0 & \text{near } c \text{ on the right} \end{cases} \text{ then } f \text{ has a local maximum at } c$$

$$\text{If } \begin{cases} f'(x) < 0 & \text{near } c \text{ on the left} \\ f'(x) > 0 & \text{near } c \text{ on the right} \end{cases} \text{ then } f \text{ has a local minimum at } c$$

If $f'(x)$ has the same sign (positive or negative) near c on <u>both sides</u> of c
then f has neither a local maximum nor a local mimimum at c

These statements should be plausible geometrically by thinking about slopes of tangent lines. In class (and in the textbook) we saw how the statements about increasing and decreasing functions are implied by the Mean Value Theorem

Example $\quad f(x) = x^2 e^{2x} \quad$ Domain $= (-\infty, \infty)$

Where is $f(x)$ increasing and decreasing? Where are the local maxima and minima?

$$f'(x) = x^2(2e^{2x}) + (2x)(e^{2x}) = 2e^{2x}(x^2 + x) = 2e^{2x}(x)(x+1)$$

$f'(x) = 0$ when $x = 0$ or -1 (the critical numbers for f)

When	$x < -1$	$-1 < x < 0$	$0 < x$
signs of e^{2x}, x, $x+1$	$+, -, -$	$+, -, +$	$+, +, +$
so	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
so $f(x)$ is	increasing	decreasing	increasing
There is		a local max at -1	local min at 0

Plotting two points: $f(0) = 0$ and $f(-1) = e^{-2} \approx 0.14$

we see that the graph looks something like: