What does f' tell us about f ?

If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing on I
If $f'(x) < 0$ on an interval I, then $f(x)$ is decreasing on I

First Derivative Test for Local Maxima and Minima

A critical number for f is a number c in the domain of f for which either $f'(c) = 0$ or $f'(c)$ does not exist. The critical numbers are the candidates for where local maxima or minima might occur.

Suppose $f(x)$ is a continuous function and that c is a critical number:

- If $\begin{cases} f'(x) > 0 & \text{near } c \text{ on the left} \\ f'(x) < 0 & \text{near } c \text{ on the right} \end{cases}$

  then f has a local maximum at c

- If $\begin{cases} f'(x) < 0 & \text{near } c \text{ on the left} \\ f'(x) > 0 & \text{near } c \text{ on the right} \end{cases}$

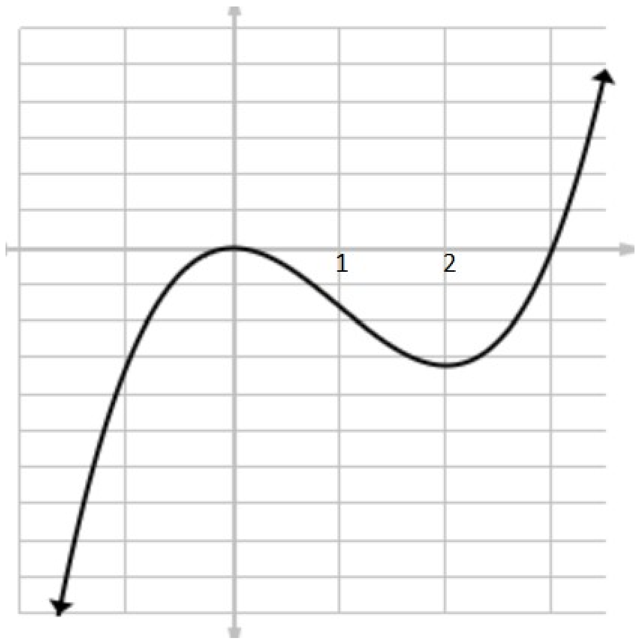
  then f has a local minimum at c

- If $f'(x)$ is positive near c on both sides of c or
  negative near c on both sides of c

  then f has neither a local maximum nor a local minimum at c

Here is the graph of $f'(x)$

Describe where $f(x)$ is increasing or decreasing and where are the local maxima and minima?



$f'(x) > 0$ when $x > 3$ so $f(x)$ is increasing when $x > 3$

$f'(x) < 0$ when $x < 0$ and when $0 < x < 3$ so $f(x)$ is decreasing when $x < 0$ and decreasing when $0 < x < 3$

$f'(x) = 0$ when $x = 0$ and when $x = 3$

Since $f'(x)$ is negative near 0 on both sides of 0, there is neither a local max no min at 0
Since $f'(x)$ is negative near 3 on the left and positive near 3 on the right, f has a local minimum at 3

From the graph (or even a formula) for $f'(x)$, we don't have enough information to reconstruct $f(x)$: there are many functions all of which have the same $f'(x)$.

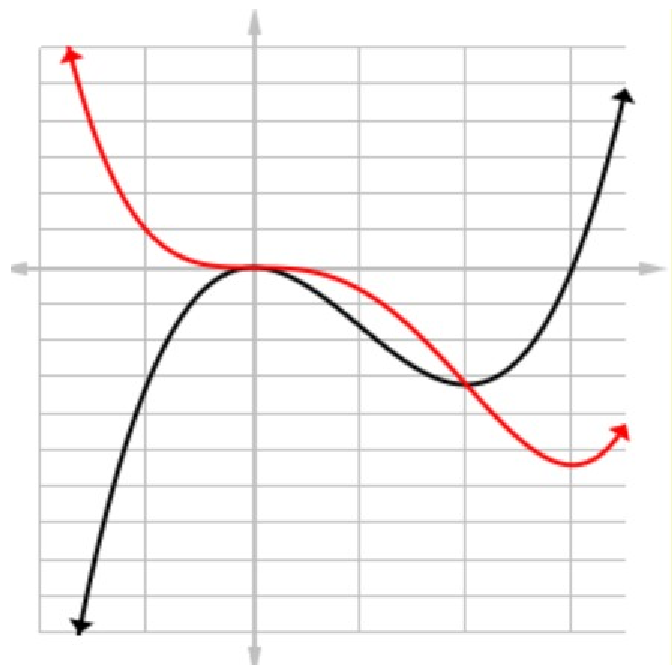
But if we assume, say, that $f(0) = 0$, we can draw a "rough" graph of f – where the shape of the graph indicates where f increasing, decreasing, and where the local max's and min's are located. The graph

The graph above was actually drawn by a graphing program, using $f(x) = x^4 - 4x^3$.

So the graph above is actually a graph of $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$.

From the formula, the critical values and $x = 0$ and $x = 3$, *just* as we deduced from the graph of $f'(x)$. And. from the formula, you can check where $f'(x) > 0$ and $f'(x) < 0$. Compare all your conclusions from those we drew above where we had only the graph of $f'(x)$ to work with.

The figure below include both the graph of $f'(x)$ and (in red) the graph of $f(x) = x^4 - 4x^3$

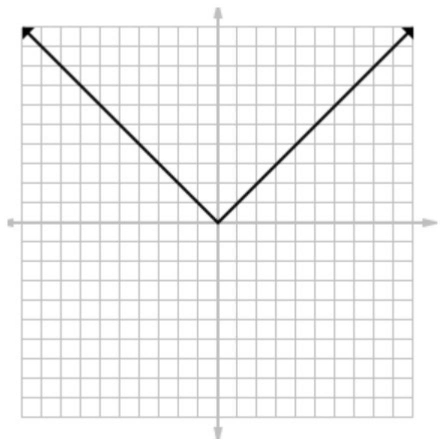


Q1 (T/F) If $y = f(x)$ has a local minimum at $x = c$, then $f'(c) = 0$.

A) True

B) False

Answer False. For example, the function $y = f(x) = |x|$ has a local minimum at $c = 0$ but $f'(0)$ does not even exist!

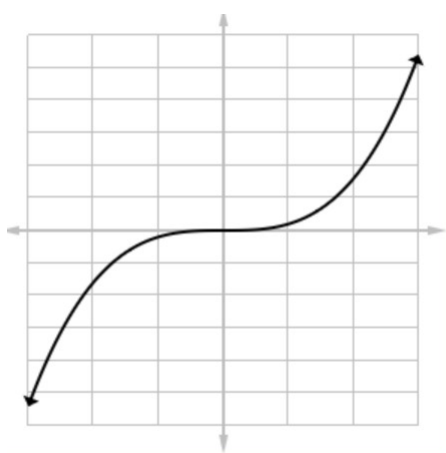


Q2 (T/F) If $y = f(x)$ and $f'(c) = 0$, then $f(x)$ has either a local maximum or a local minimum at $x = c$.

A) True

B) False

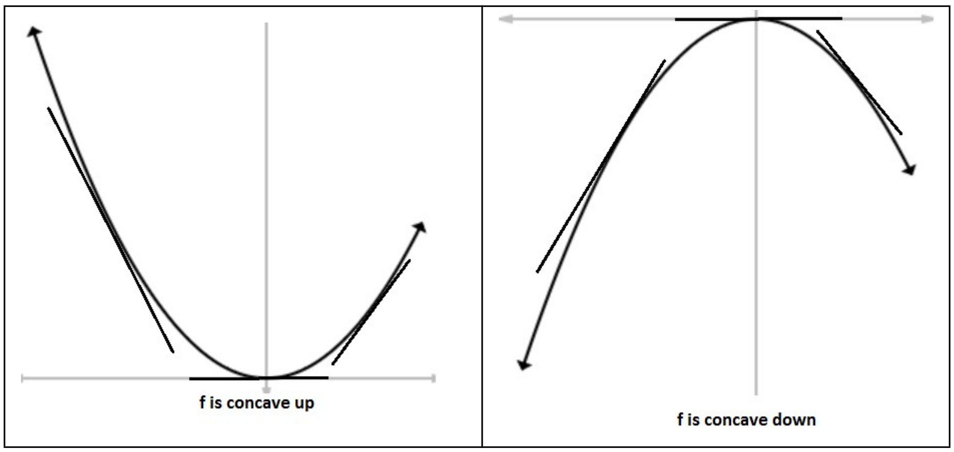
Answer False. For example, let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f'(0) = 0$. But $f(x)$ has <u>neither</u> a local maximum <u>nor</u> a local minimum at 0.



Discussed in class/textbook

f is called <u>concave up</u> on an interval I if, at every point $(x, f(x))$
the <u>graph of f lies above its tangent line</u>

f is called <u>concave down</u> on an interval I if, at every point $(x, f(x))$
the <u>graph of f lies below its tangent line</u>



Discussed in class: Suppose f is for which $f''(x)$ exists in an interval I.

The following are equivalent ways of describing "concave up" on an interval:

$f(x)$ is concave up on I
moving left to right in I, the slopes of the tangent lines are increasing
$f'(x)$ is an increasing function in I
$f''(x) > 0$ for all x in I

The following are equivalent ways of describing "concave down" on an interval:

$f(x)$ is concave down on I
moving left to right in I, the slopes of the tangent lines are decreasing
$f'(x)$ is an decreasing function in I
$f''(x) < 0$ for all x in I

We say the f has an <u>inflection point</u> at $x = c$

> if f is concave up near c on one side of c and concave down near c on the other side of c: in other words, if f has different concavity on the two sides of c (near c).
>
> This is the same as saying that $f''(x)$ has different signs on opposite sides of c (near c): positive one side, negative on the other.

Candidates for inflection points (discussed in class):

> all c in the domain where either $f''(c) = 0$ or $f''(c)$ does not exist

Example (same function as earlier): $f(x) = x^4 - 4x^3$.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

We checked earlier where f is increasing, decreasing and found the (only) local minimum at 3.

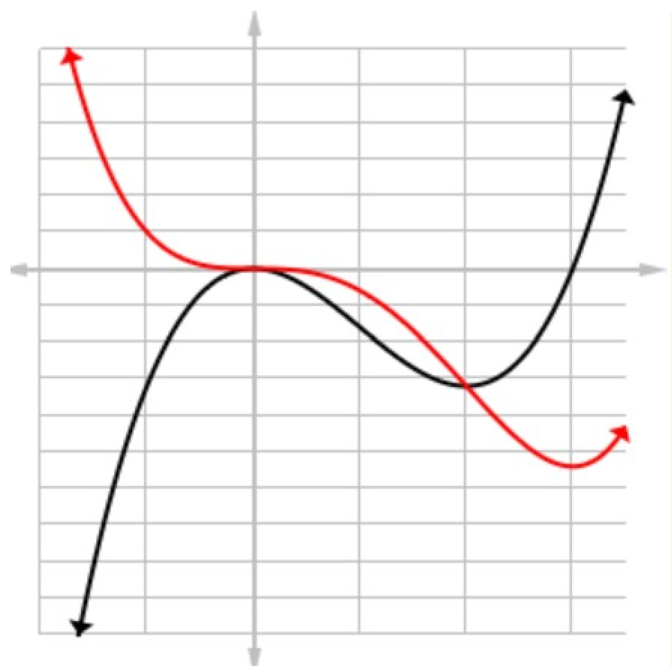
$f''(x)$ exists for every x; $f''(x) = 0$ when $x = 0$ or $x = 2$
Candidates for critical points: $0,\ 2$

Check the sign of $f''(x)$:

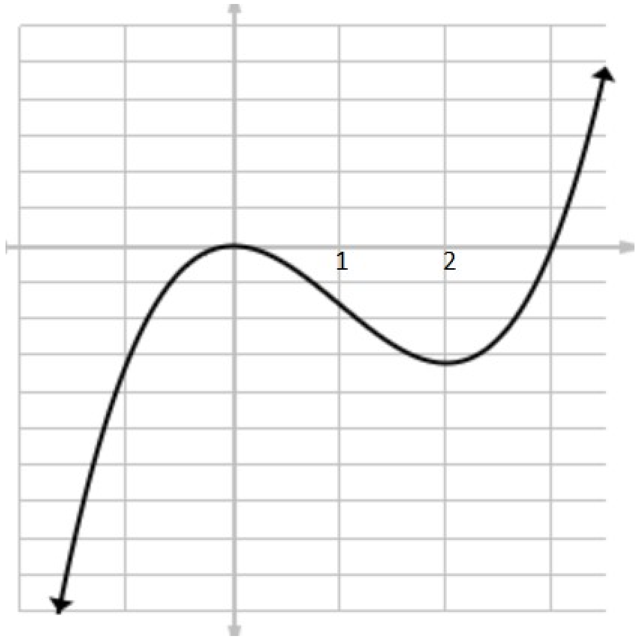
$x < 0:$	$f''(x) > 0$	(f is concave up)
$0 < x < 2:$	$f''(x) < 0$	(f is concave down)
$2 < x:$	$f''(x) > 0$	(f is concave up)

f'' changes sign at $x = 0$ and at $x = 2$ (or, equivalently, f changes concavity at $x = 0$ and $x = 2$)
so we have located two inflection points.

Spot them in the (red) graph of $f(x)$



Note: if you started with <u>only</u> the graph of $f'(x)$ and no formulas, you should still be able to spot where f has inflection points:



For $x < 0$, $f'(x)$ is increasing: so f is concave up for $x < 0$

For $0 < x < 2$, $f'(x)$ is decreasing: so f is concave down for $0 < x < 2$

For $x > 2$, $f'(x)$ is increasing: so f is concave up for $x > 2$

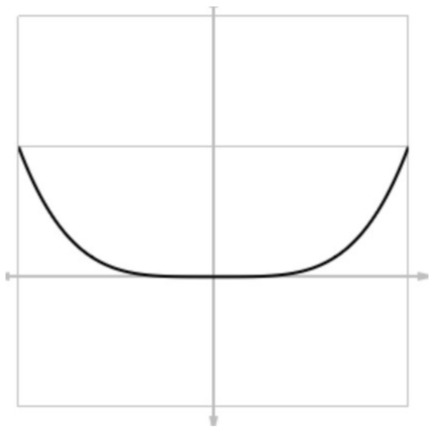
Therefore f has inflection points at 0 and 2.

Q3 (T/F) If $f''(x) = 0,$ then f has an inflection point at $c.$

A) True

B) False

Answer: false. *For* $f(x) = x^4$



$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f''(0) = 0,$ *but* $f''(x) > 0$ *both to the left and to the right of* 0 *:*
so $f(x)$ *is concave up on both sides of* 0*:* <u>*no inflection point at 0*</u>