What does f'' tell us about f ?

- If $f''(x) > 0$ on interval I, then f is concave up on I
- If $f''(x) < 0$ on interval I, then f is concave down on I

Inflection Points   Suppose f is a continuous function and c is a point in the domain of f.

If **$f''(x)$ "changes sign at c"**
(that is: f'' positive near c on one side of c and negative near c on the other side of c)

then **f has an inflection point at c**

Candidates for inflection points $=$

$=$ the critical numbers **of $f'(x)$** :

$=$ the x's for which

$(f')'(x) = f''(x) = 0$ or where

$(f')'(c) = f''(x)$ does not exist

Example Let $P(t)$ be the price ($) of the drug at time t.

$P'(t) > 0$ means that the price is rising

$P'(t) > 0$ and $P''(t) > 0$ means that the price is increasing and that $P'(t)$, the rate of increase, is increasing – that is, the price is increasing faster and faster. The graph is increasing and concave up.

$P'(t) > 0$ and $P''(t) < 0$ means that the price is increasing and that $P'(t)$, the rate of increase, is decreasing – that is, the price is increasing more and more slowly,

Figure out what it means when $\begin{cases} P'(t) < 0 \text{ and } P''(t) > 0 \\ P'(t) > 0 \text{ and } P''(t) < 0 \end{cases}$

Think about analogy between this example and the situation where f, f', and f'' represent position, velocity and acceleration.

Example $f(x) = \sin x$

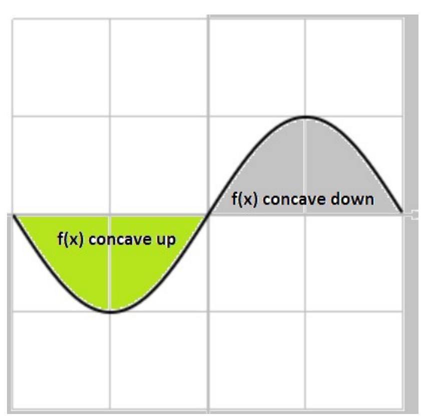
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$

$f''(x) = 0$ at $x = \ \ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

These are the candidates for inflection points.

Checking $x = 0$:

for $x < 0$ and near 0 : $f''(x) = -\sin x > 0$: f is concave up
for $x > 0$ and near 0: $f''(x) = -\sin x < 0$: f is concave down
so there is an inflection point at $x = 0$, as indicated in the graph of $\sin x$, below:



Q1 (T/F) Suppose $y = f(x)$ is continuous function on the interval $I = [7, 23]$ and that f is differentiable on $(7, 23)$.

If the absolute minimum value of f in $[7, 23]$ is 0 and the minimum occurs at two different points in $[7, 23]$, then f must have a critical number between 7 and 23.

A) True
B) False

Answer: True
There are two points in $[7, 23]$ *where an absolute minimum occurs. There must be at least one point where an absolute maximum value occurs. Of these three points, at least one must be in the "interior"* $(7, 23)$ *of the interval. It is therefore a critical number.*

Q2 If $f(x)$ is a quadratic function, then there must be a value $x = c$ where f has an inflection point.

A) Of course!
B) Certainly not!

Answer: $f(x) = ax^2 + bx + c$ *where (since the function is quadratic) the coefficient* $a \neq 0$

$$f'(x) = 2ax + b$$
$$f''(x) = 2a$$

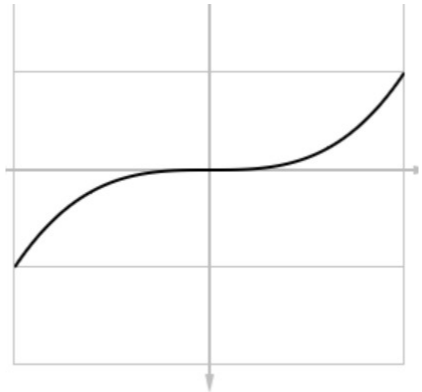
So either $f''(x) > 0$ *for all* x *(when* $a > 0$*) or* $f''(x) < 0$ *for all* x *(when* $a < 0$*). The graph of* $y = f(x)$ *is either always concave up or always concave down: no inflection points!*

(*Of course, the graph is a parabola, opening up when* $a > 0$ *or opening down when* $a < 0$)

Q3 Suppose c is in the domain of a continuous function f. It cannot happen that c is a critical number and that there is an inflection point where $x = c$.

A) True
B) False

Answer *It can happen than* c *is a critical number and that there is an inflection point at* c. *This happens, for example, when* $c = 0$ *and the function is* $y = f(x) = x^3$.



This can happen because the critical point was neither a local maximum nor a local minimum. If there were a local maximum or minimum at c, there would not also be an inflection point there.

This question is related to the Second Derivative Test for local maxima and minima.:

Suppose c is in the domain of f and f'' is continuous near c :
if $f'(c) = 0$ and $\begin{cases} f''(c) > 0 & \text{then local min at } c \\ f''(c) < 0 & \text{then local max at } c \end{cases}$

NOTE if $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$
Then

0 is a critical point

There is a local minimum at 0, but the second derivative test doesn't "pick up" on that fact. When $f''(c) = 0$, all you can say is that the Second Derivative Test leaves you undecided: the Second Derivative Test can sometimes fail (*in the sense of not giving you a decision – NOT that it ever gives you an incorrect conclusion!)*

Q4 Water is being poured at a constant rate into a conical paper cup. The height h of the liquid in the cup is a function of the volume V of water in the cup:

$$h = f(V)$$

The graph of h is

A) decreasing and concave down
B) increasing and concave up
C) increasing and concave down
D) a straight line with positive slope
E) I'm not sure.

Answer *Since* $V = \frac{1}{3}\pi r^2 h$*, we know that* $h = V/(\frac{1}{3}|r^2)$ *where r depends on* V*. So h is* *<u>not</u> a linear function of* V*, eliminating answer* D).

As V *increases,* h *also increases, so the correct answer is either* $B)$ *or* $C)$*. As water enters the cup, the remaining unfilled part of the cup is wider so as water is added at a constant rate, its depth* h *grows more slowly. The correct answer is C)*

Example Make a sketch of the graph. Here is the function and its derivatives:

$$y = \quad f(x) = \frac{x^2}{x-1}$$

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)(2(x-1))}{(x-1)^4}$$

$$= \frac{2(x-1)\Big((x-1)^2 - (x^2-2x)\Big)}{(x-1)^4}$$

$$= \frac{2(x-1)(x^2-2x+1-x^2+2x)}{(x-1)^4} = \frac{2}{(x-1)^3}$$

We will piece together several things that we know.

1) <u>From the formula for f</u>: the domain is all $x \neq 1$. Since the denominator is 0 at 1, we consider that there might be a vertical asymptote $x = 1$. To check :

$\lim_{x \to 1^+} \frac{x^2}{x-1} = \lim_{x \to 1^+} x(\frac{x}{x-1})$. As $x \to 1^+$, the denominator is positive and $\to 0$, while the numerator $x \to 1$. Therefore $\frac{x}{x-1} \to \infty = \lim_{x \to 1^+} \frac{x^2}{x-1} = \infty$

So $x = 1$ is a vertical asymptote.

Also (*similar calculation*) $\lim\limits_{x \to 1^-} \frac{x^2}{x-1} = -\infty$.

2) <u>From the derivative</u>: $f'(x) = 0$ when $x = 0,\ 2$. These are the critical numbers.
(*Notice that $x = 1$ is not a critical number: 1 is not in the domain of f*)

Since these may be "important" numbers we compute $f(0) = 0$ and $f(2) = 4$ to plot as part of the graph.

	x	$x-2$	$(x-1)^2$	$f'(x)$	$f(x)$
For $x > 2$:	$+$	$+$	$+$	$+$	increasing
For $1 < x < 2$	$+$	$-$	$+$	$-$	decreasing
For $0 < x < 1$	$+$	$-$	$+$	$-$	decreasing
For $x < 0$	$-$	$-$	$+$	$+$	increasing

Notice that we broke the interval $(0, 2)$ into two pieces:

$1 < x < 2$

$0 < x < 1$

If would be incorrect to say something about $f'(x)$ or $f(x)$ across the whole interval $(0, 2)$ since f (and therefore also f') are not defined at 1

The first derivative test shows that there is a local minimum at 2 and
a local maximum at 0

From the second derivative:

Notice that you could also use the second derivative test at 0 and 2 :

$f''(0) = \frac{2}{(0-1)^3} < 0$, so a local maximum at 0

$f''(2) = \frac{2}{(2-1)^3} > 0$, so a local minimum at 2

Since $f''(x) = \frac{2}{(x-1)^3} > 0$ when $x > 1$ f is concave up for $x > 1$

Since $f''(x) = \frac{2}{(x-1)^3} < 0$ when $x < 1$ f is concave down for $x < 1$

All these conclusions are represented in the following graph:

$y = \quad f(x) = \frac{x^2}{x-1}$