Review

What does f'' tell us about f ?

$f''(x) > 0$ on an interval I $\quad\rightarrow\quad$ f is concave up on I

$f''(x) < 0$ on an interval I $\quad\rightarrow\quad$ f is concave down on I

<u>Second Derivative Test</u> for local maxima and minima:

> Suppose c is in the domain of f and f'' is continuous near c :
>
> if $f'(c) = 0$ and
>
> $$\begin{cases} f''(c) > 0 & \text{then } f \text{ has a local min at } c \\ f''(c) < 0 & \text{then } f \text{ has a local max at } c \end{cases}$$

Q1: Consider the functions $f(x) = x^3$ and $g(x) = x^4$ Which statement is true?

A) Both functions have a local minimum at $x = 0$

B) Both functions have a local maximum at $x = 0$

C) For these functions, the Second Derivative Test does not help me decide whether the functions have a local maximum or minimum at 0

D) For these functions there is no method (except for drawing the graphs) to decide whether there is a local maximum of minimum at 0.

Answer: $f'(x) = 3x^2$ *and* $g'(x) = 4x^3$ *Note* $f'(0) = g'(0) = 0$ *so both functions have a critical number* $x = 0$.

In these two examples, we can decide what happens at $x = 0$ *by using the First Derivative test:*

$f'(x) > 0$ when $x < 0$ and also when $x > 0$, so f is increasing near 0 both on the left and right sides of 0. f has no local max or min at 0.

$g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$, so, by the First Derivative Test, g has a local minimum at 0.

But <u>both</u> $f''(0) = 0 = g''(0)$. The moral of the example is that the Second Derivative Test doesn't always work: it never gives you a wrong answer, but when the second derivative is 0 at a critical point, the test leaves you undecided.

C) is the correct answer. $f''(x) = 6x$

Q2:

In an episode of *The Simpsons* television show, Homer reads from a newspaper and announces "Here's good news! According to this eye-catching article, SAT scores are declining at a slower rate." Interpret Homer's statement in terms of a function and its first and second derivatives.

If S represents the average of all SAT scores across the country, then Homer is saying

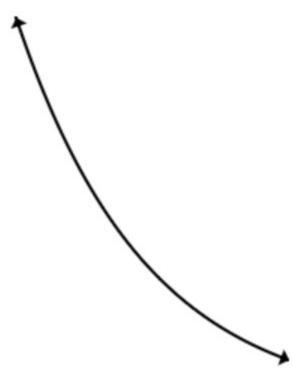
A) $S'(t) > 0$ and $S''(t) > 0$

B) $S'(t) > 0$ and $S''(t) < 0$

C) $S'(t) < 0$ and $S''(t) > 0$

D) $S'(t) < 0$ and $S''(t) < 0$

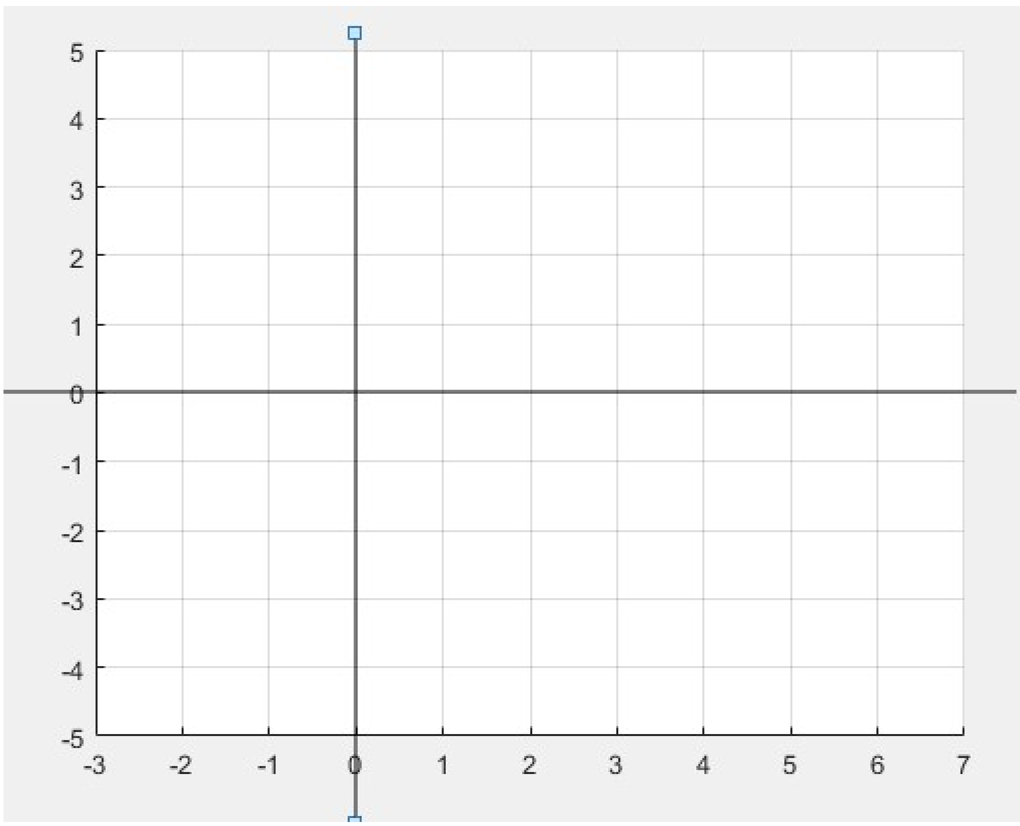
Solution: The graph of S as a function of time should be decreasing and concave up, resembling



This corresponds to answer C)

Draw the graph of a function $f(x)$ for which:

$$f'(0) = f'(2) = f'(4) = 0,$$
$$f'(x) > 0 \text{ if } x < 0 \text{ or } 2 < x < 4,$$
$$f'(x) < 0 \text{ if } 0 < x < 2 \text{ or } x > 4,$$
$$f''(x) > 0 \text{ if } 1 < x < 3, \quad f''(x) < 0 \text{ if } x < 1 \text{ or } x > 3$$



Exercise done in class.

Indeterminate Forms and L'Hospital's Rule

Find the limit:

$\lim_{x\to 0}\frac{x^2}{x}=0$ $\lim_{x\to 0^+}\frac{x}{x^2}=\infty$ $\lim_{x\to 0}\frac{2x}{x}=2$ $\lim_{x\to 0}\frac{c\,x}{x}=c$
$(\textit{where } c \textit{ is any constant})$

Looking separately at numerators and denominators, each fraction above approaches "$\frac{0}{0}$" – placed inside "quotation marks" to indicate that $\frac{0}{0}$ is not a legal arithmetic calculation. As these 4 examples show, the limit in the "$\frac{0}{0}$" case is uncertain – you need to look more carefully because the limit could turn out to be any number (or the limit might not exist at all).

"$\frac{0}{0}$" is called an indeterminate form

"$\frac{\infty}{\infty}$"

Find the limit:

$\lim_{x\to\infty}\frac{2x}{x}=2$ $\lim_{x\to\infty}\frac{cx}{x}=c$ $\lim_{x\to\infty}\frac{2x^2}{x}=\infty$ $\lim_{x\to\infty}\frac{2x}{x^2}=0$
$(\textit{where } c \textit{ is any constant})$

Looking separately at numerators and denominators, each fraction above approaches "$\frac{\infty}{\infty}$" – placed inside "quotation marks" to indicate that $\frac{\infty}{\infty}$ is not a legal arithmetic calculation (in fact ∞ isn't even a number). As these 4 examples show, the limit in the "$\frac{\infty}{\infty}$" case is uncertain – you need to look more carefully because the limit could turn out to be any number (or the limit might not exist at all)

"$\frac{\infty}{\infty}$" is called an indeterminate form" is called an indeterminate form

You can make up similar examples to show that any fractional limit with form "$\frac{\pm\infty}{\pm\infty}$" is indeterminate

Sometimes (as in the almost trivial examples above) it's easy, using some sort of algebraic simplification or manipulation of the fraction to see what the limit really is.

For example, sometimes rationalizing helps:

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\,\frac{\sqrt{x+1}\,+1}{\sqrt{x+1}\,+1}=\lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}$$

$$=\lim_{x\to 0}\ \frac{x}{x(\sqrt{x+1}+1)}=\lim_{x\to 0}\ \frac{1}{(\sqrt{x+1}+1)}=\frac{1}{2}$$

But there are other cases where its not clear how algebraic manipulation will show you the limit in the case of an indeterminate form: for example, $\lim_{x\to 1}\frac{\ln x}{x-1}$

The tool that sometimes helps is called L'Hospital's Rule (or, in a more modern French spelling: L'Hôpital's Rule)

L'Hospital's Rule

Suppose

i) $f(x)$ and $g(x)$ are differentiable near $x=a$ (not necessarily at $x=a$)

ii) $g'(x)\neq 0$ near $x=a$, and

iii) $\lim_{x\to a}\frac{f(x)}{g(x)}$ is one of the indeterminate forms "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$".

Then

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=L$ also.

Notes: 1) L'Hôpital's Rule works if "$x\to a$" is replaced by "$x\to a^+$" or "$x\to a^-$"
2) In L'Hôpital's Rule, it's OK if either a or L is $\pm\infty$.

A proof of L'Hospital's Rule is fairly complicated and involves using a more general form of the Mean Value Theorem. (See textbook, Appendix F, for a little more detail if you're interest). In the next lecture, I'll give a short discussion about why the "$\frac{0}{0}$" case of L'Hospital's Rule works.

Examples:

1) $\lim_{x\to\infty}\frac{3x^2+2x+3}{5x^2+2x+1}$. This is an indeterminate form "$\frac{\infty}{\infty}$" so L'Hospital's Rule applies:

$\lim_{x\to\infty}\frac{3x^2+2x+3}{5x^2+2x+1}=\lim_{x\to\infty}\frac{6x+2}{10x\,+\,2}$ which is still an indeterminate form "$\frac{\infty}{\infty}$". So it's

legal to use L'Hospital's Rule again

$$\lim_{x\to\infty} \frac{3x^2+2x+3}{5x^2+2x+1} = \lim_{x\to\infty} \frac{6x+2}{10x+2} = \lim_{x\to\infty} \frac{6}{10} = \frac{6}{10} = \frac{3}{5}$$

(*Notice that we didn't really* <u>*need*</u> *to use L'Hospital's Rule here. We could have just done some algebraic ma*

$$\lim_{x\to\infty} \frac{3x^2+2x+3}{5x^2+2x+1} = \lim_{x\to\infty} \frac{3+\frac{2}{x}+\frac{3}{x^2}}{5+\frac{2}{x}+\frac{1}{x^2}} = \frac{3+0+0}{5+0+0} = \frac{3}{5}.$$

2) $\lim_{x\to 1} \frac{\ln x}{(x-1)}$. This is an indeterminate form "$\frac{0}{0}$" so it's legal to use L'Hospital's Rule. (*And notice here that, unlike the preceding example, there is no obvious way to simplify or rearrange the fraction algebraically to make clear what the limit is!*)

$$\lim_{x\to 1} \frac{\ln x}{(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

3) $\lim_{x\to\infty} \frac{e^x}{x^2}$. This <u>is</u> an indeterminate form "$\frac{\infty}{\infty}$" so it's legal to use L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} \;\; (\textit{again, } \text{“}\tfrac{\infty}{\infty}\text{”}) = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

4) $\lim_{x\to 1} \frac{2x+1}{3x-1}$. This is NOT an indeterminate form: $\lim_{x\to 1} \frac{2x+1}{3x-1} = \frac{\lim_{x\to 1}(2x+1)}{\lim_{x\to 1}(3x-1)} = \frac{3}{2}$.

This calculation works using the limit laws for quotients, <u>because</u> the limit of the denominator is not 0.

If you unwisely (and "illegally") used L'Hospital's Rule, you would get an <u>**incorrect**</u> answer:

$$\lim_{x\to 1} \frac{2x+1}{3x-1} = \lim_{x\to 1} \frac{2}{3} = \frac{2}{3}$$

↑

inappropriate use of Hospitals Rule!