L'Hospital's Rule:

Suppose

that $f(x)$ and $g(x)$ are differentiable near $x = a$ (not necessarily at $x = a$ where f or g or both might not even be defined) , and

that $g'(x) \neq 0$ near $x = a$ and

that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is one of the indeterminate forms:

$$\text{“}\frac{0}{0}\text{”} \quad \text{or} \quad \text{“}\frac{\pm\infty}{\pm\infty}\text{”}$$

Then

If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ also.

Notes: 1) L'Hospital's Rule works if “$x \to a$” is replaced by “$x \to a^+$” or “$x \to a^-$”

2) In L'Hospital's Rule, it's OK if either a or L is $\pm\infty$.

Q1: Find $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$

A) 0
B) 1
C) $\frac{1}{2}$
D) 2
E) ∞

Answer $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$ is of the form "$\frac{\infty}{\infty}$" so we can try L'Hospital's Rule:
$\lim\limits_{x\to\infty} \frac{e^x}{x^2} = \lim\limits_{x\to\infty} \frac{e^x}{2x}$ (*which is still of form* $\frac{\infty}{\infty}$*, so we can try L'Hospital again*)

$= \lim\limits_{x\to\infty} \frac{e^x}{2} = \infty$

(*Note that* $\lim\limits_{x\to\infty} \frac{e^x}{2}$ *is* <u>*not*</u> *an indeterminate form: you couldn't try L'Hospital a third time*.)

The idea here intuitively, is that as $x \to \infty$, the numerator e^x of $\lim\limits_{x\to\infty} \frac{e^x}{x^2}$ is trying to make the whole fraction go to ∞, but at the same time the denominator as it $\to \infty$ is "pulling back" against the numerator and trying to make the whole fraction go to 0. In this example, the numerator "wins" – that is, "e^x goes to infinity faster that x^2."

Q2: For any $n = 1, 2, 3, \ldots$: what is $\lim\limits_{x\to\infty} \frac{e^x}{x^n}$

A) 0
B) 1
C) $\frac{1}{2}$
D) 2
E) ∞

$\lim\limits_{x\to\infty} \frac{e^x}{x^n}$ is again a "$\frac{\infty}{\infty}$" indeterminate form and we can try L'Hospital's Rule, over and over. With each application of L'Hospital's Rule, the exponent in the denominator goes down by 1, so after n applications, the denominator has become a constant and we then can see the limit. (In Q1, where $n = 2$, we got to a constant denominator after two applications of L'Hospital.)

$\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \lim\limits_{x\to\infty} \frac{e^x}{nx^{n-1}} = \lim\limits_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} =$ ...(repeating until) $= \lim\limits_{x\to\infty} \frac{e^x}{\textit{constant}} = \infty.$

Intuitively, the whole fraction $\to \infty$ because e^x grows faster than any x^n.

Example (similar to Q2)

$$\lim_{x\to\infty} \frac{e^x}{9x^2+3x-5} \quad (= \text{“}\tfrac{\infty}{\infty}\text{”}) \qquad = \lim_{x\to\infty} \frac{e^x}{18x+3} = \lim_{x\to\infty} \frac{e^x}{18} = \infty \text{ and}$$

$$\lim_{x\to\infty} \frac{e^x}{-9x^2+3x-5} \quad (= \text{“}\tfrac{\infty}{\infty}\text{”}) \qquad = \lim_{x\to\infty} \frac{e^x}{-18x+3} = \lim_{x\to\infty} \frac{e^x}{-18} = -\infty$$

In this same way you can see that $\lim_{x\to\infty} \frac{e^x}{P(x)} = \pm\infty$ when $P(x)$ is any polynomial. (The sign, $+$ or $-$, depends on the sign of the coefficient of the highest power term in $P(x)$.)

MORAL: “e^x grows faster than any polynomial” and you should be able to convince yourself easily that the same is true for any exponential function a^x (where $a > 1$).

Discussion of why L'Hospital's Rule rule works in one <u>special case</u>:

Suppose $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of form “$\frac{0}{0}$” and that $f'(x)$ and $g'(x)$ are differentiable at a

Then $f'(x)$ and $g'(x)$ are differentiable at a and also (automatically) $f(x)$ and $g(x)$ are continuous at a

Since $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of form “$\frac{0}{0}$”, we know that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

By continuity,

$$0 = \lim_{x\to a} f(x) = f(a) \quad \text{and} \qquad 0 = \lim_{x\to a} g(x) = g(a)$$

Therefore
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-0}{g(x)-0} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(x)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \underset{\uparrow}{\lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

(*because we're assuming f' and g' are continuous at a*)

Another indeterminate form "$0 \cdot \pm\infty$"

This refers to a limit $\lim\limits_{x\to a} f(x)\cdot g(x)$, where $f(x)\to 0$ and $g(x)\to \pm\infty$

The answer is indeterminate because, as $x\to a$,

> $f(x)\to 0$ is trying to make the whole product $\to 0$, while
> $g(x)\to \pm\infty$ is trying to make the whole product $\to \pm\infty$.
> f and g are "working against" each other and, without any other information, the outcome is uncertain.

For example, all of the following (simple) limits are of form "$\infty\cdot 0$" but each one has a different answer:

$$\lim_{x\to\infty}(2x)(\tfrac{1}{x}) = 2 \qquad \lim_{x\to\infty}(13x)(\tfrac{1}{x}) = 13 \qquad \lim_{x\to\infty}(2x^2)(\tfrac{1}{x}) = \infty \qquad \lim_{x\to\infty}(2x)(\tfrac{1}{x^2}) = 0$$

Since L'Hospital's Rule only applies to an indeterminate fraction limit, we rewrite a "$0\cdot \pm\infty$" limit such as $\lim\limits_{x\to a} f(x)\cdot g(x)$ in fraction form. There are two ways to do this and both lead to a fraction to which L'Hospital's Rule can be applied (*although from one problem to another, one version may be more convenient to use than the other*):

$$\lim_{x\to a} f(x)\cdot g(x) = \begin{cases} \lim\limits_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} & \text{which is a "}\tfrac{0}{0}\text{" form, or} \\ \lim\limits_{x\to a} \frac{g(x)}{\frac{1}{f(x)}} & \text{which is an "}\tfrac{\infty}{\infty}\text{" form} \end{cases}$$

Example: $\lim\limits_{x\to\infty} e^{-x}x^2$ is a "$0\cdot\infty$" indeterminate form.

$$\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

Note: You could instead try $\lim\limits_{x\to\infty} e^{-x}x^2 = \lim\limits_{x\to\infty}\frac{e^{-x}}{\frac{1}{x^2}}$ *("$\frac{0}{0}$") and use L'Hospital's Rule. But while "legal," this doesn't seem to actually work very well. Try it.*

Q3: Find $\lim\limits_{x\to 0^+} x\ln x$

A) $-\infty$
B) -1
C) $\mathbf{0}$
D) 1
E) ∞

Answer: $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-x) = 0$

Example: <u>another indeterminate form</u> "1^∞"

This refers to a limit like $\lim\limits_{x\to a} f(x)^{g(x)}$ where $f(x)\to 1$ and $g(x)\to\infty$.

This is indeterminate because

$$\begin{cases} f(x)\to 1 & \text{is trying to make the whole expression} \to 1 \\ g(x)\to\infty & \text{is trying to make the whole expression} \to \infty \end{cases}$$

It's impossible to tell, without more work, what the limit really is in a specific case. For example

$\lim\limits_{x\to 0^+} (3^x)^{1/x}$ and $\lim\limits_{x\to 0^+} (5^x)^{1/x}$ are both of form "1^∞"

but $\lim\limits_{x\to 0^+} (3^x)^{1/x} = \lim\limits_{x\to 0^+} 3 = 3$, and $\lim\limits_{x\to 0^+} (5^x)^{1/x} = \lim\limits_{x\to 0^+} 5 = 5$

Example: $\lim\limits_{x\to 0} (2x+1)^{1/x}$ (this is of the form "1^∞")

Let $y = (2x+1)^{1/x}$
so $\ln y = \ln (2x+1)^{1/x} = \frac{1}{x}\ln(2x+1) = \frac{\ln(2x+1)}{x}$

$\lim\limits_{x\to 0} (2x+1)^{1/x} = \lim\limits_{x\to 0} \frac{\ln(2x+1)}{x}$ ("0/0") $= \lim\limits_{x\to 0} \frac{\frac{2}{2x+1}}{1} = 2$

so as $x\to 0$, $\ln y \to 2$

so $$\begin{array}{l} e^{\ln y} \to e^2 \\ \| \\ y \end{array}$$

Therefore $\lim\limits_{x\to 0} (2x+1)^{1/x} = e^2$