**L'Hospital's Rule**:

Suppose

$f(x)$ and $g(x)$ are differentiable near $x = a$ (not necessarily at $x = a$)

$g'(x) \neq 0$ near $x = a$

$\lim_{x \to a} \frac{f(x)}{g(x)}$ is one of the indeterminate forms "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$".

Then

If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ also.

*Notes: 1) L'Hospital's Rule works if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$"*

*2) In L'Hospital's Rule, it's OK if either $a$ or $L$ is $\pm\infty$.*

## More examples of indeterminate forms

Fractions "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

Previously worked examples: $\lim\limits_{x\to\infty} \frac{3x^2+2x+3}{5x^2+2x+1}$ $\lim\limits_{x\to 1} \frac{\ln x}{(x-1)}$

(Use L'Hospital's Rule directly)

Products "$0 \cdot \pm\infty$"

Previously worked example: $\lim\limits_{x\to\infty} x^2 e^{-x}$

Rewrite as a fraction to use L'Hospital's Rule: $\lim\limits_{x\to\infty} \frac{x^2}{e^x}$

Indeterminate forms involving exponentiation

Previously worked examples:

"$1^\infty$" $\lim\limits_{x\to 0^+} (2x+1)^{1/x}$

"$\infty^0$" $\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$ (*in discussion section*)

"$\frac{0}{0}$" $\lim\limits_{x\to 0^+} x^x$ (*in textbook)*

Use logarithms to "get rid of" the exponent: see examples

Indeterminate form "$\infty - \infty$"

To illustrate: "$\infty - \infty$"

$$\lim_{x\to\infty} (x+7) - x = 7$$
$$\lim_{x\to\infty} (x-33) - x = -33$$
$$\lim_{x\to\infty} (x+c) - x = c$$
$$\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1) = \infty$$

All of these limits have form "$\infty - \infty$" but the limit can come out quite differently: for "$\infty - \infty$" you can't say <u>automatically</u> what the limit will be. That's why such forms are called indeterminate.

Note: a form like "$\infty + \infty$" i <u>not</u> indeterminate: why not?

Example: $\lim_{x\to1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ is form "$\infty - \infty$": rewrite as a fraction to try to use L'Hospital's Rule (combine over a common denominator):

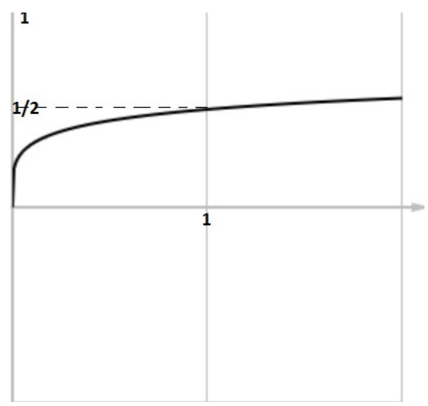
$$\lim_{x\to1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to1^+}\left(\frac{x\ln x-(x-1)}{(x-1)\ln x}\right) \quad (\textit{which is form } \text{``}\tfrac{0}{0}\text{''})$$

$$= \lim_{x\to1^+}\frac{(x\cdot\frac{1}{x}+\ln(x)-1)}{(x-1)\cdot\frac{1}{x}+\ln x} = \lim_{x\to1^+}\frac{\ln x}{(x-1)\cdot\frac{1}{x}+\ln x} \quad (\textit{which is form } \text{``}\tfrac{0}{0}\text{''} \textit{ again})$$

$$= \lim_{x\to1^+}\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}} = \frac{1}{2}.$$

(For visual confirmation, here's the graph:

$$f(x) = \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$$




EXERCISE: How is $\lim_{x\to1^-}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ different ?

How is $\lim_{x\to1^+}\left(\frac{x}{x-1} + \frac{1}{\ln x}\right)$ different?

Example: $\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x}$ ("$\infty - \infty$")

<u>Method 1 (L'Hospital)</u> Rewrite as a fraction to try to use L'Hospital's Rule:

$$\sqrt{x+1} - \sqrt{x} = \sqrt{x}\left(\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x}}\right) = \sqrt{x}\left(\sqrt{\frac{x+1}{x}}-1\right) = \frac{\sqrt{\frac{x+1}{x}}-1}{\frac{1}{\sqrt{x}}}$$

so

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{\frac{x+1}{x}}-1}{\frac{1}{\sqrt{x}}} \qquad (\text{“}\tfrac{0}{0}\text{”})$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{\frac{x+1}{x}}} \cdot (-\frac{1}{x^2})}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{\frac{x+1}{x}}} \cdot (-\frac{1}{x^2})}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{\frac{x+1}{x}}}(-x^{-2})(-2x^{\frac{3}{2}}) = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x+1}{x}}}x^{-\frac{1}{2}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x+1}{x}}} \cdot \frac{1}{\sqrt{x}} = 1 \cdot 0 = 0$$

<u>Method 2 (rationalize)</u>

$$\lim_{x\to\infty} \sqrt{x+1} - \sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x+1}-\sqrt{x}}{1} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0$$

Moral: sometimes not using L'Hospital's Rule is better!

**Caution: don't think something is an indeterminate form when it isn't!**

For example, "$0^{\infty}$" is <u>not</u> indeterminate:

Suppose $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$

Then for $\lim\limits_{x\to a} f(x)^{g(x)}$ (form "$\frac{0}{\infty}$")

$f(x) \to 0$ is trying to make $f(x)^{g(x)} \to 0$
As $f(x)$ gets close to 0, the exponent $g(x)$, which is growing larger and larger, <u>helps</u> to make $f(x)^{g(x)} \to 0$ : in other words, $f$ and $g$ are "cooperating" (rather than competing) to make $\lim\limits_{x\to a} f(x)^{g(x)} = 0$

You can see this numerically in the following specific example: $\lim\limits_{x\to\infty} (\frac{1}{x})^x$. Just calculate a few values:

| $x$ | $f(x) = (\frac{1}{x})^x$ |
|---|---|
| 1 | $f(1) = 1^1 = 1$ |
| 5 | $f(5) = (\frac{1}{5})^5 = \frac{1}{3125}$ |
| 10 | $f(10) = (\frac{1}{10})^{10} = (10^{-1})^{10} = 10^{-10}$ |
| 100 | $f(100) = (\frac{1}{100})^{100} = (10^{-2})^{100} = 10^{-200}$ |
| $\downarrow$ | $\downarrow$ |
| $\infty$ | 0 |

Here are a few other forms that are <u>not</u> indeterminate:

"$\infty \cdot \infty$" "$\frac{\infty}{0}$" "$\frac{0}{\infty}$" "$0^{\infty}$" "$\infty^{\infty}$"

Q1: Consider $\lim\limits_{x \to 0} \frac{5^x - 4^x}{3^x - 2^x}$.

**A) the limit exists and is positive**
B) the limit is 0
C) the limit exists and is negative
D) the limit doesn't exist ( $= \infty$)
E) the limit doesn't exist ( $= -\infty$)

*Answer: The limit has form "$\frac{0}{0}$". Since* $\frac{d}{dx} a^x = (\ln a) a^x$.

$$\lim_{x \to 0} \frac{5^x - 4^x}{3^x - 2^x} = \lim_{x \to 0} \frac{\ln(5)5^x - \ln(4)4^x}{\ln(3)\, 3^x - \ln(2)\, 2^x} = \frac{\ln(5) - \ln(4)}{\ln(3) - \ln(2)}.$$

Since the function $\ln x$ is <u>increasing,</u> $\ln(5) > \ln(4)$ and $\ln(3) > \ln(2)$.
so $\frac{\ln(5) - \ln(4)}{\ln(3) - \ln(2)} > 0$

Q2: Find $\lim\limits_{x \to \infty} \frac{\ln x}{x^p}$, where the constant $p > 0$.

You can then conclude that

**A) As $x \to \infty$, $\ln x$ grows much slower than $\sqrt{x}$**
B) As $x \to \infty$, $\ln x$ is approximately proportional to $\sqrt{x}$
C) As $x \to \infty$, $\ln x$ grows much faster than $\sqrt{x}$

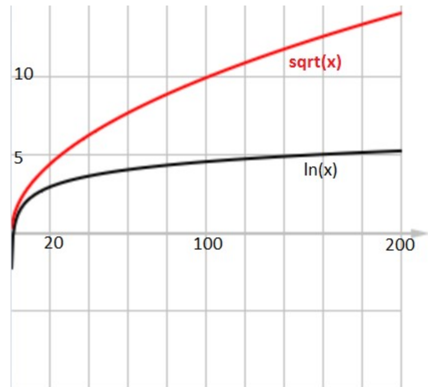
$\lim\limits_{x \to \infty} \frac{\ln x}{x^p}$ in an indeterminate form "$\frac{\infty}{\infty}$". Using L'Hospital's Rule,

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = \lim_{x \to \infty} \frac{\frac{1}{x}}{px^{p-1}} = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{1}{px^{p-1}} = \lim_{x \to \infty} \frac{1}{px^p} = 0 \text{ (since denominator} \to \infty)$$

When $p = \frac{1}{2}$, in particular, $\lim\limits_{x \to \infty} \frac{\ln x}{x^p} = \lim\limits_{x \to \infty} \frac{\ln x}{\sqrt{x}} = 0$.

Both numerator and denominator $\to \infty$ but $\frac{\ln x}{\sqrt{x}} \to 0$. So it must be the numerator goes toward infinity more slowly than the denominator (denominator "wins" and makes the fraction $\to$ 0)

This is illustrated in the following graph (*note the scale on the $x$-axis*)

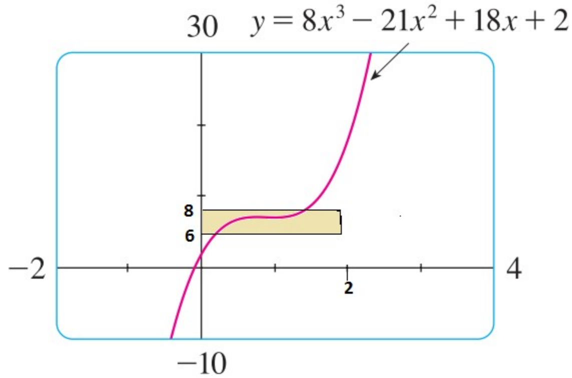



<u>The moral is</u>: As $x \to \infty$, $\ln x$ grows more slowly that any poser $x^p$ (where $p > 0$)

Pulling all sorts of things together in curve sketching:

Why is calculus important?

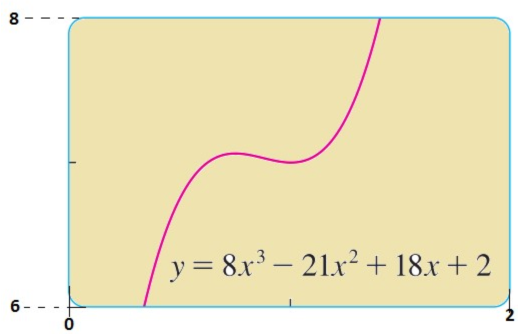
In this graph, taken from Stewart's textbook, you think you see how the graph behaves.




However the viewing window for the calculator/computer was scaled so that some "more delicate" features of the graph aren't visible.

If you compute derivatives, you find easily that there is a local max at $x = \frac{3}{4}$, a local min at $x = 1$, and an inflection point at $x = \frac{7}{8}$.

Of course, you can see these if you rescale you viewing window:




But you can never be sure, using a computer/calculator view, that you've used a "good"-sized window – that is, you can never be sure, with your computer alone, that you haven't missed some important features of the graph.

Calculus will let you find "delicate" features of the graph that you might otherwise miss. Ideally, calculus and computing should complement each other.