## Curve sketching

$y=f(x)=\left(\frac{1}{x}\right)^{x}$
Domain: let's agree just $x>0$ : there are many negative $x$ values for which $f(x)$ is

$$
\text { undefined, for example } f\left(-\frac{1}{2}\right)=(-2)^{-\frac{1}{2}}=\frac{1}{\sqrt{-2}} ? ? ?
$$

Note $f(x)>0$ always
Intercepts? $\quad y$ is never 0 , so graph has no $x$-intercepts;
$\underline{\text { Is there a vertical asymptote where }} x=0$ ? look at $\lim _{x \rightarrow 0^{+}}\left(\frac{1}{x}\right)^{x} \quad\left(=" \infty^{0 "}\right)$

$$
\begin{aligned}
& y=\ln \left(\frac{1}{x}\right)^{x}=x \ln \left(\frac{1}{x}\right)=x(\ln 1-\ln x)=-x \ln x \\
& \text { so } \lim _{x \rightarrow 0^{+}} \ln y=\lim _{x \rightarrow 0^{+}}\left(\frac{1}{x}\right)^{x}=-\lim _{x \rightarrow 0^{+}} x \ln x \quad(0 \cdot \infty)=-\lim _{x \rightarrow 0^{+}} \frac{\ln x}{\frac{1}{x}} \quad\left(" \frac{-\infty}{\infty}\right. \text { ") } \\
& =-\lim _{x \rightarrow 0^{+}} \frac{\frac{1}{x}}{-\frac{1}{x^{2}}}=\lim _{x \rightarrow 0^{+}} x=0
\end{aligned}
$$

Therefore $\ln y \rightarrow 0$, so $y=e^{\ln y} \rightarrow e^{0}=1$
$\lim _{x \rightarrow 0^{+}}\left(\frac{1}{x}\right)^{x}=1:$ no vertical asymptote at $x=0$

## Is there a horizontal asymptote?

Is there a vertical asymptote at $x=0 ? \quad$ Is there a horizontal asymptote?
Check $\lim y=\lim _{x \rightarrow \infty}\left(\frac{1}{x}\right)^{x}=0: \quad(" \underline{0} "$ - $\underline{\text { not }}$ indeterminate! $)$
A horizontal asymptote: $y=0$
(Since the domain only contains positive $x^{\prime}$ s, we don't also check $\lim _{x \rightarrow-\infty}\left(\frac{1}{x}\right)^{x}$.)
Local maxima or minima? intervals of increase or decrease?

$$
\begin{aligned}
& y=f(x)=\left(\frac{1}{x}\right)^{x} \quad \text { Use logarithmic differentiation } \\
& \ln y=\ln \left(\frac{1}{x}\right)^{x}=x \ln \left(\frac{1}{x}\right)=x(\ln 1-\ln x)=-x \ln x \\
& \frac{y^{\prime}}{y}=-\left(x\left(\frac{1}{x}\right)+\ln x\right)=-(1+\ln x) \\
& y^{\prime}=y(-(1+\ln x))=-\left(\frac{1}{x}\right)^{x}(1+\ln x)=f^{\prime}(x)
\end{aligned}
$$

$$
\begin{aligned}
& f^{\prime}(x)=0 \text { when } 1+\ln x=0 \\
& \begin{aligned}
\ln x & =-1 \\
x=e^{\ln x} & =e^{-1}=\frac{1}{e} \approx 0.37
\end{aligned} \\
& \text { for } 0<x<\frac{1}{e}: \quad-\left(\frac{1}{x}\right)^{x}<0 \quad(1+\ln x)<0 \quad f^{\prime}(x)>0 \quad f \text { inc } \\
& \text { for } \frac{1}{e}<x: \quad-\left(\frac{1}{x}\right)^{x}<0 \quad(1+\ln x)>0 \quad f^{\prime}(x)<0 \quad f \mathrm{dec}
\end{aligned}
$$

$f$ has a local maximum at $x=\frac{1}{e}$, where $f(1 / e)=e^{1 / e} \approx 1.44$
The second derivative is not that difficult to calculate if you don't get frightened:

$$
f^{\prime}(x)=-\left(\frac{1}{x}\right)^{x}(1+\ln x)
$$

so

$$
\begin{aligned}
f^{\prime \prime}(x) & =\left(-\left(\frac{1}{x}\right)^{x}\right) \frac{d}{d x}(1+\ln x)+(1+\ln x) \frac{d}{d x}\left(-\left(\frac{1}{x}\right)^{x}\right) \\
& =\left(-\left(\frac{1}{x}\right)^{x}\right)\left(\frac{1}{x}\right) \quad+(1+\ln x)\left(-\left(-\left(\frac{1}{x}\right)^{x}(1+\ln x)\right)\right. \\
& =\left(\frac{1}{x}\right)^{x}\left((1+\ln x)^{2}-\frac{1}{x}\right)
\end{aligned}
$$

To solve $f^{\prime \prime}(x)=0$, we'd set $(1+\ln x)^{2}-\frac{1}{x}=0$, but there's no way to actually solve this equation algebraically. So usually we'd either turn to an computer equation solver for assistance or give up on locating inflections points. However, in this example, we might notice, by good luck, that since $\ln 1=0, x=1$ satisfies the equation $f^{\prime \prime}(x)=0$ :

$$
(1+\ln 1)^{2}-\frac{1}{1}=0
$$

So, in this example, "inspection" gives us a candidate for an inflection point. Testing this candidate is manageable:

How, without a calculator? $\left(\frac{1}{x}\right)^{x}$ is always positive; so the sign of $f^{\prime \prime}(x)$ is the same as the sign of $(1+\ln x)^{2}-\frac{1}{x}$

For $x=1:(1+\ln x)^{2}-\frac{1}{x}=0$
Since $\ln 1=0$ and $\ln x$ is an increasing function,

$$
\begin{aligned}
& \text { when } x>1,(1+\ln x)^{2}>1 \text { and } \frac{1}{x}<1 \text { so } \\
&(1+\ln x)^{2}-\frac{1}{x}>0 \\
& \text { and therefore } f(x) \text { is concave up } \\
& \text { when } x<1,(1+\ln x)^{2}<1 \text { and } \frac{1}{x}>1 \text { so } \\
&(1+\ln x)^{2}-\frac{1}{x}<0 \\
& \text { and therefore } f(x) \text { is concave down }
\end{aligned}
$$

Since $f$ changes concavity at $x=1$, there is an inflection point on the graph at $(1, f(1))=\left(1,\left(\frac{1}{1}\right)^{1}\right)=(1,1)$.

The graph is shown below. Check that all the information we found is accurately reflected in the graph!


Q1: Which graph is $y=f(x)=\frac{x^{2}+1}{x^{4}+2}$ ?


Answer: i) from the formula, $f(0)=\frac{1}{2}$, and that only fits ( $A$ )
or
ii) from the formula, $f(x)>0$ for all $x$, so the answer must be (A) or (C) also $\lim _{x \rightarrow \infty} f(x)=0$ so $y=0$ is a horizontal asymptote. That looks to be not the case in ( $C$ ), so the answer must be (A)
iii) $f(-x)=\frac{\frac{o r}{(-x)^{2}+1}}{(-x)^{4}+2}=\frac{x^{2}+1}{x^{4}+2}=f(x)$. This means the graph must be symmetric across the y-axis (see illustration below). Only (A) has this symmetry property.


Q2: Which graph (above) is $f(x)=\frac{2 x}{x^{2}+1}$ ?

$\underline{o r}$
$f(-x)=\frac{2(-x)}{(-x)^{2}+1}=-\frac{2 x}{x^{2}+1}=-f(x)$ so the graph must be symmetric with respect to the origin (see illustration below) Only D) has this property.


QUESTION: What functions (not pictured) have the property that their graphs are both symmetric with respect to the $y$-axis and symmetric with respect to the origin.
(Note that, for example, the curve $x^{2}+y^{2}=1$ has both these symmetries, but that curve is not the graph of a function.

Example: Sketch the graph of $y=f(x)=\frac{x^{2}-4}{x^{2}+4}$
Domain $=(-\infty, \infty)=\mathbb{R}$ (notation for the set of all real numbers)
? Intercepts: $y=0$ when $x= \pm 2 ; x=0$ when $y=\frac{0-4}{0+4}-1$.
? Symmetry across the $y$ axis: is $f(x)=\frac{x^{2}-4}{x^{2}+4}=\frac{(-x)^{2}-4}{(-x)^{2}+4}=f(-x)$ ? Yes.
? Symmetry across the origin: is $f(x)=\frac{x^{2}-4}{x^{2}+4}=-\frac{(-x)^{2}-4}{(-x)^{2}+4}=-f(-x)$ ? No
? Horizontal asymptotes:

$$
\begin{aligned}
& \lim _{x \rightarrow \infty} \frac{x^{2}-4}{x^{2}+4} \stackrel{\text { LH }}{=} \lim _{x \rightarrow \infty} \frac{2 x}{2 x}=1 \text { so } y=1 \text { is a horizontal asymptote. } \\
& \lim _{x \rightarrow-\infty} \frac{x^{2}-4}{x^{2}+4} \stackrel{\text { LH }}{=} \lim _{x \rightarrow-\infty} \frac{2 x}{2 x}=1 \text { so } y=1 \text { (again, as } x \rightarrow-\infty \text { ) }
\end{aligned}
$$

? Vertical asymptotes: none because $\lim _{x \rightarrow a} f(x)=\frac{a 2-4}{a^{2}+4}$ (never $\pm \infty$ ) for every $a$
? Intervals of increase/decrease? local maxima or minima :

$$
\begin{aligned}
& y^{\prime}=f^{\prime}(x)=\frac{\left(x^{2}+4\right)(2 x)-\left(x^{2}-4\right)(2 x)}{\left(x^{2}+4\right)^{2}}=\frac{2 x\left(\left(x^{2}+4\right)-\left(x^{2}-4\right)\right)}{\left(x^{2}+4\right)^{2}}=\frac{16 x}{\left(x^{2}+4\right)^{2}} \\
& f^{\prime}(x)=0 \text { when } x=0 \\
& \quad \text { for } x<0: \quad f^{\prime}(x)<0 \quad f \text { is decreasing } \\
& \text { for } x>0: \quad f^{\prime}(x)>0 \quad f \text { is increasing } \\
& \\
& \text { so } f \text { has a local minimum at } x=0 . f(0)=\frac{0-4}{0+4}=-1 \\
& y^{\prime \prime}=f^{\prime \prime}(x)=\frac{16\left(x^{2}+4\right)^{2}-16 x\left(2\left(x^{2}+4\right)(2 x)\right)}{\left(x^{2}+4\right)^{4}}=\frac{16\left(x^{2}+4\right)\left[\left(x^{2}+4\right)-x(2)(2 x)\right]}{\left(x^{2}+4\right)^{4}} \\
& =\frac{16\left(4-3 x^{2}\right)}{\left(x^{2}+4\right)^{3}}=0 \text { when } 4-3 x^{2}=0, \text { that is, when } x= \pm \sqrt{\frac{4}{3}}= \pm \frac{2}{\sqrt{3}}
\end{aligned}
$$

Since the denominator of $f^{\prime \prime}(x)>0$ for all $x$, the sign of $f^{\prime \prime}(x)$ is the same as the sign of $4-3 x^{2}$ :

$$
\begin{array}{lll}
\text { for } x<-\frac{2}{\sqrt{3}} & 4-3 x^{2}<0 & f \text { concave down } \\
\text { for }-\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}} & 4-3 x^{2}>0 & f \text { concave up } \\
\text { for } \frac{2}{\sqrt{3}}<x & 4-3 x^{2}<0 & f \text { concave down }
\end{array}
$$

So $f$ has inflection points at $\pm \frac{2}{\sqrt{3}} \approx 1.15$, where $f\left( \pm \frac{2}{\sqrt{3}}\right)=\frac{\left(\frac{2}{\sqrt{3}}\right)^{2}-4}{\left(\frac{2}{\sqrt{3}}\right)^{2}+4}=-\frac{1}{2}$
All this information fits together as in the graph below:


