Example: Sketch the graph of $y = f(x) = \frac{\sin x}{1+\cos x}$

<u>Domain</u> All x except where $\cos x = -1$: so $x \neq \ldots, -3\pi, -\pi, \pi, 3\pi$,
(*that is,* $x \neq$ *odd multiple of* π)

Notice that $f(x)$ has period 2π : $f(x+2\pi) = \frac{\sin(x+2\pi)}{1+\cos(x+2\pi)} = \frac{\sin x}{1+\cos(x)} = f(x)$.
Since the function "repeats itself" every 2π, we can focus our attention on any interval of length 2π, draw the graph there, and then just keep repeating our graph in adjacent intervals of length 2π to the left and right.

So we will focus on $f(x)$ for the interval $(-\pi, \pi)$, remembering that f is not even defined at $\pm\pi$.

<u>? Intercepts</u>: <u>in the interval</u> $(-\pi, \pi)$:

y intercept: when $x = 0$, then $y = 0$
x intercept: in this interval, $\sin x = 0$ only when $x = 0$. Therefore
$y = \frac{\sin x}{1+\cos x}$ 0 where $x = 0$

The graph crosses the x-axis and the y-axis only at $(0, 0)$

<u>? Symmetry with respect to the y axis or with respect to the origin</u>

$$f(-x) = \frac{\sin(-x)}{1+\cos(-x)} = -\frac{\sin x}{1+\cos x} = -f(x)$$

so the graph is symmetric with respect to the origin

<u>? Horizontal and vertical asymptotes</u>: since we are restricting our attention to the interval $(-\pi, \pi)$, we can't compute $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$. No horizontal asymptotes

For a in $(-\pi, \pi)$, $\lim\limits_{x\to a} f(x) = \geq \frac{\sin a}{1+\cos a}$ (denominator is never 0). The only possibilities for vertical asymptotes would be at the endpoints: check at π and $-\pi$.

$$\lim_{x\to\pi^-} \frac{\sin x}{1+\cos x} \left(\text{“}\tfrac{0}{0}\text{”}\right) \overset{\text{LH}}{=} \lim_{x\to\pi^-} -\frac{\cos x}{\sin x} = \text{“} -\frac{\text{negative}, \ \to -1}{\text{positive}, \ \to 0} \text{”} = \infty$$

$$\lim_{x\to-\pi^+} \frac{\sin x}{1+\cos x} \overset{\text{LH}}{=} \lim_{x\to-\pi^+} -\frac{\cos x}{\sin x} = \text{“} -\frac{\text{negative}, \ \to -1}{\text{negative}, \ \to 0} \text{”} = -\infty$$

(Note: once you know that $\lim\limits_{x\to\pi^-} \frac{\sin x}{1+\cos x} = \infty$*, then you could immediately decide that* $\lim\limits_{x\to-\pi^+} \frac{\sin x}{1+\cos x} = -\infty$ *by using the fact the graph is symmetric with respect to the origin.*)

? <u>Intervals of increase/decrease, local maxima or minima</u> :

$$y' = \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{(1+\cos x)}$$

Always, $1 + \cos x \geq 0$ and in the interval $(-\pi, \pi)$, $1 + \cos x > 0$
Therefore $y' > 0$ everywhere in $(-\pi, \pi)$: the function is always increasing in this interval. There are no critical points and no local maxima or minima.

For graphing purposes, notice that $f'(0) = \frac{1}{2}$, so as the graph passes through $(0, 0)$ it should have slope $\frac{1}{2}$.

? <u>Concavity and Inflection points</u> :

$$y'' = \frac{(1+\cos x)(0) - (1)(-\sin x)}{(1+\cos x)^2} = \frac{\sin x}{(1+\cos x)^2}$$

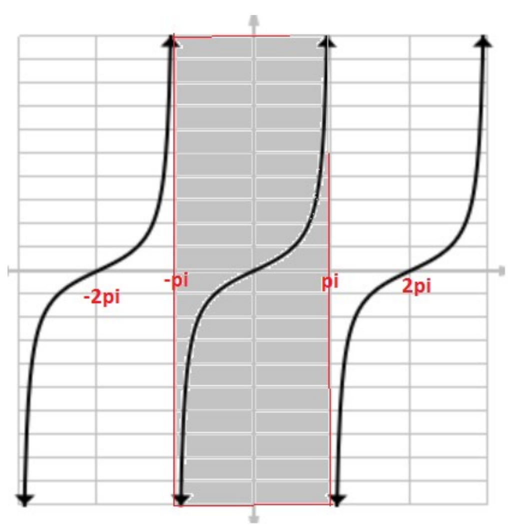
In the interval $(-\pi, \pi)$, $y'' = \frac{\sin x}{(1+\cos x)^2} = 0$ where $\sin x = 0$, that is at $x = 0$.

In $(-\pi, \pi)$, the denominator is always , and $\begin{cases} \text{for } -\pi < x < 0 : & \sin x < 0 \\ \text{for } 0 < x < \pi : & \sin x > 0 \end{cases}$

so $\begin{cases} \text{for } -\pi < x < 0 : & y'' < 0 & f \text{ is concave down} \\ \text{for } 0 < x < \pi : & y'' > 0 & f \text{ is concave up} \end{cases}$

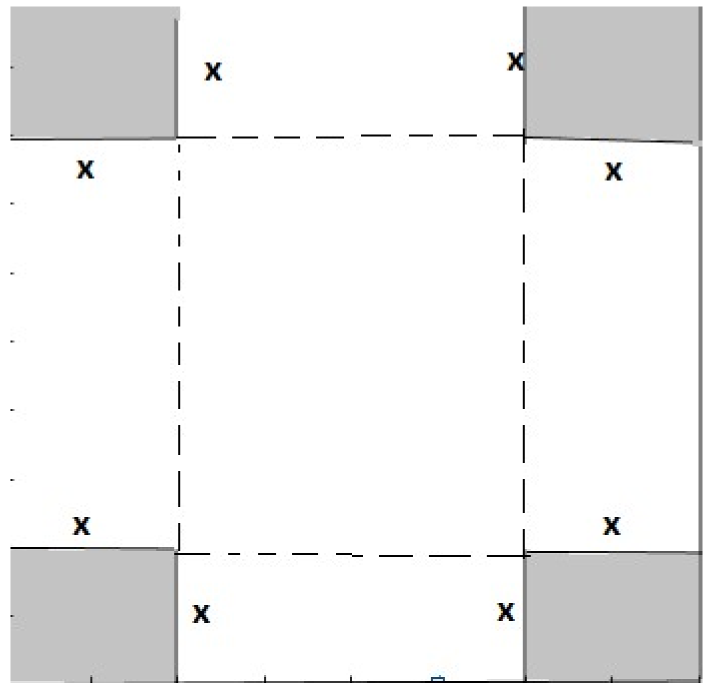
and there is an inflection point at 0.

<u>All this information is gathered together in the graph below. We worked only in the interval $(-\pi, \pi)$, shaded gray. Then repeat the graph in each adjacent intervals of width 2π.</u>



Example: (*Optimization*)

A rectangular box (open on the top – no lid!) is to be constructed from a 3ft x 3ft square of cardboard by cutting a small square out of each corner and folding up along the dashed lines to form the sides of the box. What is the largest possible volume for the box?



$\leftarrow$ 3 $\rightarrow$

Q1: If an x x x square is cut from each corner of the cardboard, what will be the volume of the resulting box?

A) $V = (3 - x)x^2$
B) $V = (3 - 2x)^2 x$
C) $V = (3 - x)^2 x$
D) $V = (3 - 2x)x^2$
E) $V = (3 - 2x)^3 x^3$

Answer: The rectangle surrounded by the dashed lines becomes the bottom of the box: it has area $(3 - 2x)(3 - 2x)$. When the sides are folded up, the height of the box will be x. So $V = (3 - 2x)^2 x$

Q2: What values for x make sense in this problem?

A) $0 \leq x \leq 3$
B) $0 \leq x \leq 2$
C) $0 \leq x \leq \frac{3}{2}$
D) $0 < x < \frac{3}{2}$
E) $0 < x < 3$

Answer: We could argue between

C) $0 \leq x \leq \frac{3}{2}$ *and*
D) $0 < x < \frac{3}{2}$,

Since the cardboard is $3x3$ *we certainly can't cut out square corners where* $x > \frac{3}{2}$ *(the cardboard isn't lig enough) nor squares with a "negative side"* x.

The issue is whether to allow $x = 0$ *(no squares removed, no sides to fold up – a "box" with height* $= 0$ *and therefore* $V = 0$*) and whether to allow* $x = \frac{3}{2}$ *(after removing the corner squares, no cardboard is left, and we could think of that too as producing a "box" with* $V = 0$*)*

But is doesn't really matter: either way, we are looking for the *<u>largest volume possible</u>,* *and the answer for how to do that doesn't depend whether we decide count* $V = 0$ *as the "smallest" possible volume)*

I prefer thinking of $0 \leq x \leq \frac{3}{2}$ *because then* $v(x)$ *is a continuous function on a closed interval* $[0, \frac{3}{2}]$ *and the* *<u>Extreme Value Theorem</u>* *guarantees, that there* *<u>will be</u>* *an absolute maximum and minimum V for these* x*'s, and it makes finding them simpler.*

Notice (just in case you were dubious) that changing x <u>does</u> change the volume V of the box you create. Here's a table with just a couple of sample calculations:

x	V
0	0
0.25	1.5625
0.5	2
0.75	1.6875
1	1
1.25	0.3125
1.5	0

We want to find the absolute maximum value of $V = (3-2x)^2 x$ on the interval $[0, \frac{3}{2}]$

There must be an absolute maximum and minimum value for V : these could occur at the endpoints at any critical points in $(0, \frac{3}{2})$.

$V' \quad = 2(3-2x)(-2)x + (3-2x)^2(1)$
$\quad = (3-2x)[-4x + (3-2x)] = (3-2x)(3-6x) = 0$
so $x = \frac{3}{2}$ or $x = \frac{1}{2}$

The only critical number in $(0, \frac{3}{2})$ is $\frac{1}{2}$.

Test the value of V for each candidate: $V(0) = V(\frac{3}{2}) = 0$
$V(\frac{1}{2}) = 2^2(\frac{1}{2}) = 2$

So the absolute maximum volume is 2 (ft^3), obtained when the squares cut from the corners are $\frac{1}{2}$ x $\frac{1}{2}$. (*As observed before we began calculating,* $x = 0 = \frac{3}{2}$ *produce "boxes" with absolute minimum volume* $V = 0$.

Same Example, continued: What if we had decided to work with
$V = (x)(3-2x)^2 x$ on the interval $(0, \frac{3}{2})$ (excluding the possibility of "boxes with volume 0)?

This turns out to be a little more involved. We are not guaranteed mathematically that the continuous function $V(x)$ will have an absolute maximum value: we have to compute to see whether it does or not.

Just as above: setting $V' = 0$ gives that $x = \frac{1}{2}$ is the only critical point in the interval $(0, \frac{3}{2})$.

Now we look at the sign of the derivative:

in $(0, \frac{3}{2})$	$(3-2x) > 0$	
	$(6-3x) > 0$	when $0 < x < \frac{1}{2}$
	$(6-3x) < 0$	when $\frac{1}{2} < x < \frac{3}{2}$
Therefore	$f'(x) > 0$	for all $x < \frac{1}{2}$ in $(0, \frac{3}{2})$
	$f'(x) < 0$	for all $x > \frac{1}{2}$ in $(0, \frac{3}{2})$

This means that there must be an absolute maximum at $\frac{1}{2}$, and $V(\frac{1}{2}) = 2$, as before.

Notice that on the interval $(0, \frac{3}{2})$ *the is no absolute minimum value for V* *(not that we care in this problem).. If you think that 0 is still the absolute minimum value for v, I ask you: at what x in $(0, \frac{3}{2})$ does it occur?)*

This solution uses the "First Derivative Test for Absolute Maxima and Minima" as stated on p. 333 of the text:

> **First Derivative Test for Absolute Extreme Values** Suppose that c is a critical number of a continuous function f defined on an interval.
>
> (a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
>
> (b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

We have been using a "First Derivative Test" to identify local maxima and minima, and in general, it doesn't help with absolute maxima and minima. But if f is continuous on an interval I and c is a critical number in I and, say,

$f'(x) > 0$ for ALL x in I, $x < c$ and
$f'(x) < 0$ for ALL x in I, $x > c$

then (of course) f has a local maximum at c but also an absolute maximum at c

The key word up above is ALL: it implies that there are no critical points other than c in I *(if d is any other point in the interval I, then either $d < c$ or $d > c$: so either $f'(d) > 0$ or $f'(d) < 0$ – and therefore d isn't a critical number.* The function is rising (increasing) at ALL x's $< c$ in I and then falling for ALL x's $> c$ in I: so the value $(c, f(c))$ is the highest point on the graph.

Similarly, $f'(x) < 0$ for ALL x in I, $x < c$ and
$f'(x) > 0$ for ALL x in I, $x > c$

tells you that $f(x)$ its absolute minimum value in I at $x = c$.

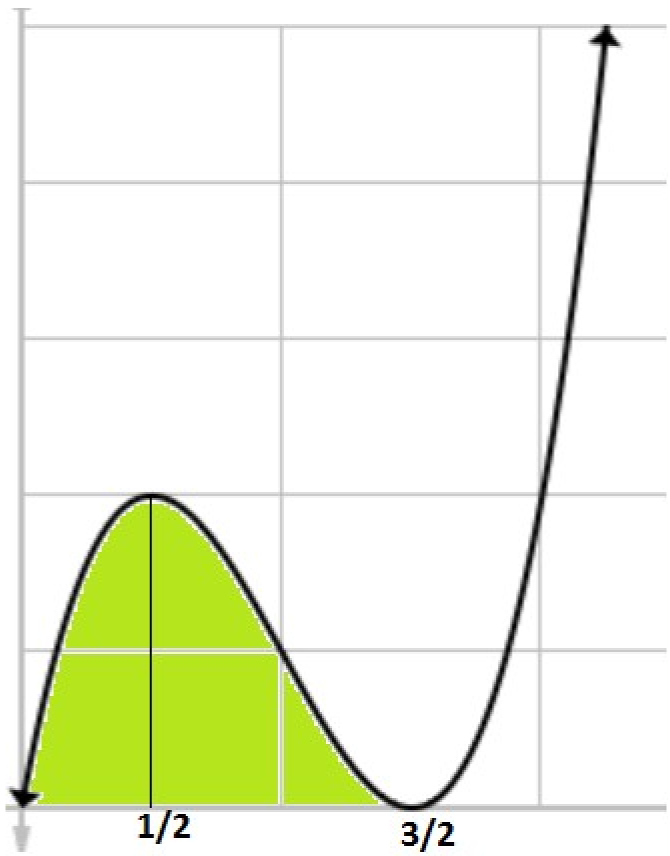
Look at the following graph:

$V = (3-2x)^2 x$ where $0 \le x \le \frac{3}{2}$
(*shaded green*)

Our optimization problem restricted x $[0.\frac{3}{2}]$ or $(0, \frac{3}{2})$

Either way, the part of the graph we were looking at shows an absolute maximum value 2 at $x = \frac{1}{2}$. On $(0, \frac{3}{2})$ the First Derivative test detects the absolute max at $\frac{1}{2}$ because

in $I = (0, \frac{3}{2})$ $f'(x) > 0$ for all $x < \frac{1}{2}$ and $f'(x) < 0$ for all $> \frac{1}{2}$



But if, in some other situation, we were looking a $V(x)$ on, say, the interval $I = (0, 3)$ then the same check of the first derivative would reveal that 2 is a local maximum value but 2 is not an absolute maximum value:

$f'(x)$ < 0 is NOT true for ALL $x > \frac{1}{2}$ in the new interval $(0, 3)$