Suppose $F(x)$ is defined on an interval $I$ :
$F(x)$ is called an antiderivative (abbreviate: $A D$ ) of $f(x)$ if $F^{\prime}(x)=f(x)$

$$
F^{\prime}(x)=f(x)
$$

$$
\begin{array}{llc}
F(x) & \stackrel{\text { derivative }}{\longleftrightarrow} & F^{\prime}(x)=f(x) \\
F(x) & \text { antiderivative } & f(x)
\end{array}
$$

Q1: One of the following is $f(x)$ and the other is an antiderivative $F(x)$. Which is which?

A) Graph A is $f$ and $\quad$ Graph B is $F$
B) Graph A is $F$ and $\quad$ Graph B is $f$

Answer: Suppose Graph B) is $F(x)$. At 4, the tangent line to $F(x)$ has slope $\approx 0$, so $F^{\prime}(4) \approx 0$. But Graph A) at 4 does not have value $\approx 0$. So Graph A) cannot be the derivative of Graph B.

So it must be that Graph $A)=F(x)$ (the antiderivative) and Graph B) $=F^{\prime}(x)=f(x)$

Each differentiation formula gives an antidifferentiation formula.
$f(x) \quad$ An antiderivative $F(x)$

| 2 | $2 x$ | (Check by differentiating) |
| :---: | :---: | :---: |
| $k$ (constant) | $k x$ | (Check by differentiating) |
| $x$ | $\frac{1}{2} x^{2}$ | (Check by differentiating!) |
| $k x$ | $k\left(\frac{1}{2} x^{2}\right)=\frac{k}{2} x^{2}$ | (Check by differentiating) |
| $x^{2}$ | $\frac{1}{3} x^{3}$ | (Check by differentiating) |
| $\cos x$ | $\sin x$ | (Check by differentiating) |
| $\sin x$ | $-\cos x$ | (Check by differentiating) |
| $e^{x}$ | $e^{x}$ | (Check by differentiating) |
| $\frac{1}{1+x^{2}}$ | $\arctan x$ | (Check by differentiating) |
| $x^{n}$ | $\frac{1}{n+1} x^{n+1}$ provided $n \neq-1$ (Check by differentiating) |  |
| $x^{-1}=\frac{1}{x}$ | $\ln x$ (on any interval of positive numbers, for example, ( $0, \infty$ ) ) |  |
|  | Notice that if $F$ | $\ln \|x\|= \begin{cases}\ln (x) & \text { if } x>0 \\ \ln (-x) & \text { if } x<0\end{cases}$ |
|  | then | $F^{\prime}(x)=\left\{\begin{array}{l} \frac{1}{x} \\ \frac{1}{(-x)}(-1)=\frac{1}{x} \end{array}\right.$ |
|  |  | $\begin{gathered} =\frac{1}{x} \quad(\text { on any interval not } \\ \text { containing } 0) \end{gathered}$ |

$x^{-1}=\frac{1}{x}$
$\ln x \xrightarrow{\text { HOW??? }}$

So we kind write a broader formula in the list:
$\ln |x| \quad$ (on any interval not containing 0 )
$x \ln x-x$

Here, ii is certainly not clear how someone might discover this AD, but nevertheless you can verify that it is correct by differentiating:

$$
\frac{d}{d x}(x \ln x-x)=x\left(\frac{1}{x}\right)+(1) \ln x-1=\ln x .
$$

From earlier in the course:
On an interval $I$ : if $F^{\prime}(x)=G^{\prime}(x)$, then $F(x)$ and $G(x)$ must differ by a constant, that is

$$
F(x)=G(x)+C \quad \text { (where, of course, } C \text { might be negative }
$$

If $F(x)$ and $G(x)$ are both antiderivatives for $f(x)$ on the interval $I$, then

$$
\left.\begin{array}{rl}
F^{\prime}(x) & =G^{\prime}(x) \quad(\text { both }=f(x)) \\
\text { so } & F(x)
\end{array}\right)=G(x)+C \text {. }
$$

so (excerpting from the previous table):

| $f(x)$ | an antiderivative | all antiderivatives <br> (or the "most general antiderivative <br> formula") on an interval |
| :--- | :--- | :--- |
| $x^{2}$ | $\frac{1}{3} x^{3}$ | $\frac{1}{3} x^{3}+C$ |
| $\cos x$ | $\sin x$ | $\sin x+C$ |
| $\sin x$ | $-\cos x$ | $-\cos x+C$ |
| $e^{x}$ | $e^{x}$ | $e^{x}+C$ |
| $\frac{1}{1+x^{2}}$ | $\arctan x$ | $\arctan x+C$ |
|  | etc. |  |

Q2: On the interval $\mathbb{R}=(-\infty, \infty)$ : Which of the following is an antiderivative for $f(x)=3 x^{2}\left(x^{3}+1\right)^{7}$ ?
A) $F(x)=(x+1)^{8}$
B) $F(x)=(x+1)^{8}-11$
C) $F(x)=63 x^{4}\left(x^{3}+1\right)^{6}$
D) $F(x)=\frac{1}{8}\left(x^{3}+1\right)^{8}$
E) $F(x)=\frac{1}{8}\left(x^{3}+1\right)^{8}-11$

Answers: both D) and E) are correct since

$$
\frac{d}{d x}\left(\frac{1}{8}\left(x^{3}+1\right)^{8}\right)=\frac{1}{8} \cdot 8\left(x^{3}+1\right)^{7}\left(3 x^{2}\right)=3 x^{2}\left(x^{3}+1\right)^{7}
$$

Q3: On the interval $\mathbb{R}=(-\infty, \infty)$ : Which of the following is an antiderivative for $f(x)=\frac{e^{x}}{1+e^{x}}$ ?
A) $F(x)=\frac{1+e^{x}}{e^{x}}$
B) $F(x)=\frac{1+e^{x}}{e^{x}}+13$
C) $F(x)=\ln \left(e^{x}\right)+3$
D) $F(x)=\ln \left(\frac{1+e^{x}}{e^{x}}\right)$
E) $F(x)=\ln \left(1+e^{x}\right)+7$

Answer: e) since $F^{\prime}(x)=\frac{d}{d x}\left(\ln \left(1+e^{x}\right)+7\right)=\frac{1}{1+e^{x}}\left(e^{x}\right)+0=\frac{e^{x}}{1+e^{x}}$.
Just for information (compare Q1):


The lower graph is $f(x)=\frac{e^{x}}{1+e^{x}}$, and the upper is $F(x)=\ln \left(1+\mathrm{e}^{x}\right)+\frac{1}{2}$

For motion along a straight line,

$$
\begin{array}{ll}
s & =\text { position }=f(t) \\
\frac{d s}{d t}=f^{\prime}(t) & =\text { velocity }=v(t) \\
\frac{d^{2} s}{d t^{2}}=\frac{d v}{d t}=v^{\prime}(t) & =\text { acceleration }=a(t)
\end{array}
$$

So the velocity is an antiderivative of the acceleration and the position is an antiderivative of the velocity

Example: An arrow is shot straight up from the ground, leaving the bow with an initial velocity of $320 \mathrm{ft} / \mathrm{sec}$. How high does it go? When does it return to the ground? What is its velocity when it hits the ground?

Assume the only force acting is gravity (in particular, no air resistance).
Let $s=0$ denote ground level, and let "up" be the positive direction.


Then at time $t=0: s=0 \mathrm{ft}$ and $v=320 \mathrm{ft} / \mathrm{sec}$
The acceleration due to gravity $=-32 \quad \mathrm{ft} / \mathrm{sec}^{2}$; in metric units, this is usually taken to be $-9.8 \mathrm{~m} / \mathrm{sec}^{2}$; The sign for $a$ is - since gravity pulls in the down (negative) direction
$a=-32 \quad$ so (antiderivative!)
$v=-32 t+C . \quad$ But since $320=v(0)=-32(0)+C$, we must have $c=320$, so $v=-32 t+320$.

So (antiderivative!0
$s=-32\left(\frac{1}{2} t^{2}\right)+320 t+D . \quad$ But since $0=s(0)=-32\left(\frac{1}{2}(0)^{2}\right)+320(0)+D$, we must have $D=0$, so
$s=-16 t^{2}+320 t$

To find the maximum value ("height") for $s$, we set the derivative
$\frac{d s}{d t}=v=-32 t+320=0$, and get $t=10(\mathrm{sec})$
What is $s$ when $t=10 ? s(10)=-16(10)^{2}+320(10) 1600 \mathrm{ft}$. This is how high the arrow goes.

The arrow is a ground level when $s=-16 t^{2}+320 t=-16 t(t-20)=0$, that is, when $t=0$ or $t=20$ : the start and end of the arrow's flight. (Note: time $u p=$ time down $=10 \mathrm{sec}$ ).

What is its velocity when it hits the ground? $v(20)=-32(20)+320=-320 \mathrm{ft} / \mathrm{sec}$. (Notice that this is the same speed but opposite direction as when $t=0$.)

All of this information about the arrow's behavior comes from calculus + the one physical measurement that $a=-32 \mathrm{ft} / \mathrm{sec}^{2}$.

Example: A car is moving along the highway at a constant velocity of $55 \mathrm{ft} / \mathrm{sec} \quad(=37.5$ mph ). Suddenly the brakes are applied, producing a constant deceleration of $15 \mathrm{ft} / \mathrm{sec}^{2}$. How far does the car travel before stopping?

Call $t=0$ the time at which the brakes are applied, and start measuring position from that moment also - so $s=0$ when $t=0$ : $s$ tells you how far you've traveled measuring from the place where the brakes were applied.

The velocity is positive and applying the brakes produces an opposite acceleration:
$a(t)=-15$ (constant) (the " - " sign is also implied by the "de" in the word decelerated: the velocity is decreasing so its derivative, the acceleration, is negative.)
$v(t)=-15 t+C$ and since $v(0)=55$, we must have $v(t)=-15 t+55$, and therefore $s(t)=-15\left(\frac{1}{2} t^{2}\right)+55 t+D$. But $s(0)=0$, so $D$ must be 0.

The car comes to a stop when $v=0=-15 t+55$, so when $t=\frac{55}{15}=\frac{11}{3} \mathrm{sec}$.
Therefore the stopping position )(distance traveled after hitting the brakes) is

$$
s\left(\frac{11}{3}\right)=-\frac{15}{2}\left(\frac{11}{3}\right)^{2}+55\left(\frac{11}{3}\right) \approx 100.83 \mathrm{ft} .
$$

(Note: there's always some "reaction time" between the decision/instinct to hit the brake and actually doing it. If there were, say, a 1 second reaction time here, that would add an additional 55 feet to the stopping distance.)

