Review from precalculus: Sigma (Σ) notation is a way to efficiently write down long sums.

$$a_1 + a_2 + a_3 + ... + a_n = \sum_{i=1}^{n} a_i$$

The notation simply means that, in turn, for each i, find a_i : then add them all together.

Example: $S = \sum_{i=1}^{3} (2i + 3) = 5 + 7 + 9 = 21$

Example: $S = \sum_{i=1}^{n} i = 1 + 2 + ... + n$ sum of the first n natural numbers)

We can find a nice formula for S : write the sum down twice, in opposite orders:

$$\begin{array}{llllllll} S = 1 & + 2 & + 3 & + & ... & + n - 1 & + n \\ S = n & + (n-1) & + n - 2) & + & ... & + \quad 2 & + 1 \end{array}$$

Add to get

$$2S = (n+1) \quad + (n+1) \quad + (n+1) \quad\quad + (n+1) \quad + (n+1)$$

$$= n(n+1)$$

so $S = \frac{n(n+1)}{2}$

Example: $S = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2$ (the sum of the squares of the first n natural numbers)

There is also a nice formula for this sum: $S = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(*See the extra sheet "Sums of Squares" to derive this formula and more.*

Example: $\sum_{i=2}^{6} 7$. Here,$a_i = 7$ (for every i, <u>starting</u> with $i = 2$)

$$\sum_{i=2}^{6} 7 = a_2 + a_3 + a_4 + a_5 + a_6 = 7 + 7 + 7 + 7 + 7 = 35$$

Q1: $\sum_{i=1}^{3} i(i+1) =$

A) $\sum_{k=6}^{8} k(k+1)$

B) $\sum_{k=6}^{8} (k+5)(k+6)$

C) $\sum_{k=6}^{8} (k-5)(k-4)$

D) $\sum_{k=2}^{4} (k-1)(k-2)$

E) $\sum_{k=2}^{4} (k+1)(k+2)$

Answer C:

$$\begin{aligned}\sum_{i=1}^{3} i(i+1) &= (1)(2) + (2)(3) + 3(4) \\ &= \sum_{k=6}^{8} (k-5)(k-4)\end{aligned}$$

Note that the same sum can be written in different ways in sigma notation. Try writing this same sum in a few other ways.

We know how to find areas of regions bounded by straight line segments: rectangles, triangles, polygons (cut up into non-overlapping triangles). We also know how to find the area of one kind of region with a "curved" boundary: a circle with radius r has area $a = \pi r^2$.

Finding areas of more general regions with "curved boundaries" is one motivation for a new idea. We approximate the area as the sum of the area of n rectangles. The approximation improves as n gets larger, and the limit of the approximations is the exact area. (*This is the style as we used to find the slope of a tangent line to* $y = f(x)$ *at* $(a, f(a))$: *make an approximation using the slope of a secant line through* $(a, f(a)$ *and a nearby point* $(f, (x))$. *The secant line has slope* $\frac{f(x) - f(a)}{x - a}$ *and the limit of the approximations* $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ *gives the exact slope of the tangent line, assuming this limit exists.*

To be more precise, we will work on this problem: suppose $y = f(x)$ is continuous on the interval $[a, b]$ and that $f(x) \geq 0$ (*so that the graph is always above the* x*-axis*). What is the area A under the graph and above the interval $[a, b]$?

1) <u>Approximate the area A</u> (refer to the picture, from the textbook, below):

a) Pick n $(= 1, 2, 3...)$

b) Subdivide interval $[a, b]$ into n <u>equal</u> subintervals. So each subinterval will have length $\Delta x = \frac{b-a}{n}$

Call the endpoints of these subintervals

$$\begin{array}{cccc} x_0, & x_1, \ldots.., & x_{n-1}, & x_n \\ \| & & & \| \\ a & & & b \end{array}$$

c) Inside each subinterval, <u>choose</u> a "sample point" x_i^*
(*how? the choice is up to you; you could choose each* x_i^* = *the right endpoint of the subinterval; or the left endpoint; or the midpoint, or ...*)

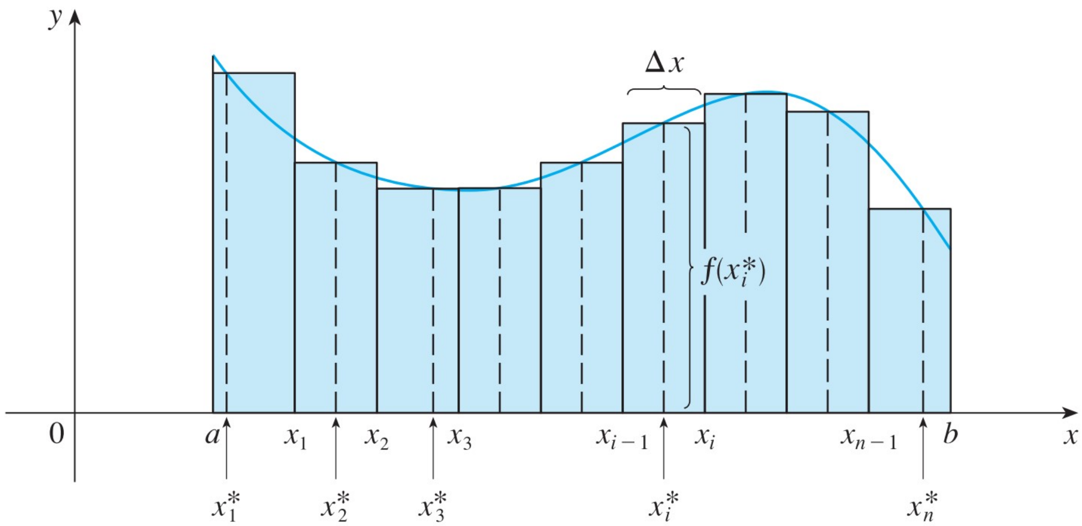
d) For each subinterval: compute $f(x_i^*)$
Using the subinterval as base, make a rectangle whose height is $f(x_i^*)$
The area of this rectangle is (height)(base) $= f(x_i^*)\Delta x$

e) Sum up the areas of all these rectangles:

$$S_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

S_n is called a <u>Riemann Sum</u>

The value of S_n depends on n and on how the x_i^*'s are chosen. S_n <u>approximates the area A</u>



2) As you make n larger and larger, S_n gives a better and better approximation to the exact value of the area A.

3) The limit of the approximations gives the exact area:

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = A = \text{the exact area under the}$$

graph. (*It turns out that for "nice" functions – for example, when f is continuous – the limit will always exist and the value of the limit* <u>*doesn't depend*</u> *on how the x_i^*'s were chosen*)

In class, we followed the steps above for the function $y = f(x) = x^2$ over the interval $[0, 1]$. These calculations are done in detail, with illustration, in the textbook (pp. 366-369 so they aren't reproduced again in these notes except to summarize:

R_n denotes the Riemann sum with n equal subintervals and where the right endpoints are chosen as the sample points. L_n denotes the Riemann sum with n equal subintervals and where the left endpoints are chosen as the sample points.

For example,

$$R_4 = \overset{f(\frac{1}{4})}{(\tfrac{1}{4})^2}\overset{\Delta x}{(\tfrac{1}{4})} + (\tfrac{2}{4})^2(\tfrac{1}{4}) + (\tfrac{3}{4})^2(\tfrac{1}{4}) + (\tfrac{4}{4})^2(\tfrac{1}{4})$$

In this particular example, the function $f(x) = x^2$ is increasing so using right endpoints to create the rectangle heights produces rectangles that are "tool tall" – they stick out above the graph of f. Similarly, using left endpoints produces rectangles that are "too short" – totally under the graph of f. So R_n (in this example) always overestimates area A and L_n always underestimates A.

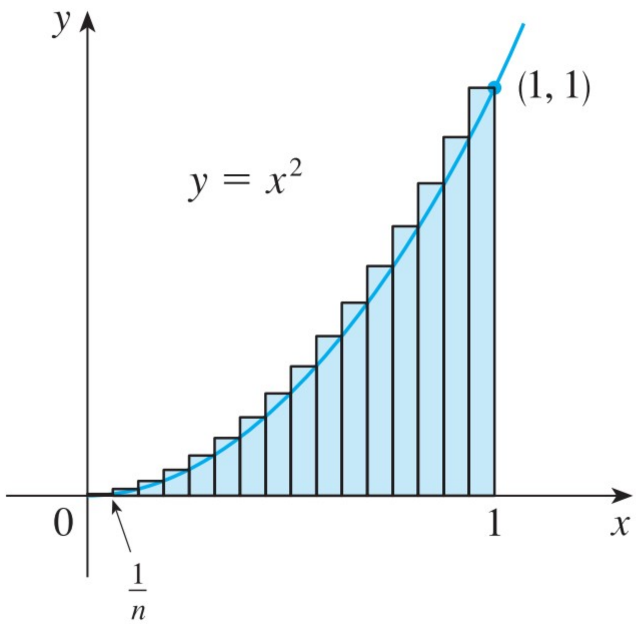
The following table shows the calculated values of both L_n and R_n for some larger value of n. The roe for $n = 1000$ tells us that

$$0.3328335 < A < .3338335$$

The table suggests the guess that as $n \to \infty$, $L_n \to \frac{1}{3}$ and $R_n \to \frac{1}{3}$ and that therefore $A = \frac{1}{3}$ (exact!)

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

We can verify this (at does the text) by writing out and simplifying an exact formula for R_n (doable here because this is such a simple example):



The subintervals have width $\Delta x = \frac{1}{n}$ and the right endpoints (sample points in the subintervals) are $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}(=1)$.

$$
\begin{aligned}
R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad &= f(\tfrac{1}{n})\Delta x + f(\tfrac{2}{n})\Delta x + f(\tfrac{3}{n})\Delta x + \dots + f(\tfrac{n}{n})\Delta x \\
&= (\tfrac{1}{n})^2\Delta x + (\tfrac{2}{n})^2\Delta x + (\tfrac{3}{n})^2\Delta x + \dots + (\tfrac{n}{n})^2\Delta x \\
&= (\tfrac{1}{n})^2(\tfrac{1}{n}) + (\tfrac{2}{n})^2(\tfrac{1}{n}) + (\tfrac{3}{n})^2(\tfrac{1}{n}) + \dots + (\tfrac{n}{n})^2(\tfrac{1}{n}) \\
&= \tfrac{1}{n^3}(1^2 + 2^2 + 3^2 + \dots + n^2) \\
&= \tfrac{1}{n^3}\left(\tfrac{n(n+1)(2n+1)}{6}\right) = \tfrac{2n^3+3n^2+n}{6n^3}
\end{aligned}
$$

So the (exact area) A under the graph is $A \quad = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6} = \frac{2}{6} = \frac{1}{3} \text{ (as we guessed).}$$

This was a fair amount of work for a very simple example. We will see that there's a much easier way to deal with more complicated areas.