Review from last lecture: Suppose $f(x)$ is continuous on $[a,b]$, $f(x) \geq 0$.

Divide $[a,b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$.

Pick "sample points" $x_1^*,\ x_2^*, \ldots x_n^*$ in the subintervals.

Form the Riemann sum

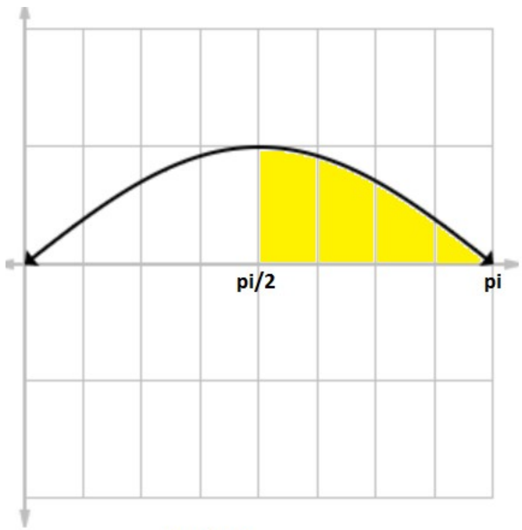
$$S = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\lim_{n\to\infty} S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \text{area } A \text{ under graph and above } [a,b]$$

Because f is continuous, it can be shown that the limit will exist and always be the same no matter how the x_is are chosen. So, for example, if the x_i^*'s are the right or left endpoints, then we get the same value for A:

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = A$$

Q1: Approximate the area A under the graph of $y = \sin x$ and above the interval $[\frac{\pi}{2}.\ \pi]$ using $n = 4$ and choosing right endpoints as sample points in each subinterval.



The Riemann sum looks like

$$R_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

What is Δx and what is x_3^*

A) $\Delta x = \frac{\pi}{4} \quad x_3^* = \frac{6\pi}{8}$ B) $\Delta x = \frac{\pi}{4} \quad x_3^* = \frac{7\pi}{8}$

C) $\Delta x = \frac{\pi}{8} \quad x_3^* = \frac{6\pi}{8}$ D) $\Delta x = \frac{\pi}{8} \quad x_3^* = \frac{7\pi}{8}$

E) $\Delta x = \frac{\pi}{2} \quad x_3^* = \frac{7\pi}{8}$

Answer D) : $\Delta x = \frac{\pi - \frac{\pi}{2}}{4} = \frac{\pi}{8}$

In the yellow shaded area, the interval $[\frac{\pi}{2}, \pi]$ *is divided into 4 subintervals of width* $\frac{\pi}{8}$. *The right endpoints are* $\frac{5\pi}{8},\ \frac{6\pi}{8},\ \frac{7\pi}{8},\ \frac{8\pi}{8} = \pi$
The third right endpoint is $x_3^* = \frac{7\pi}{8}$

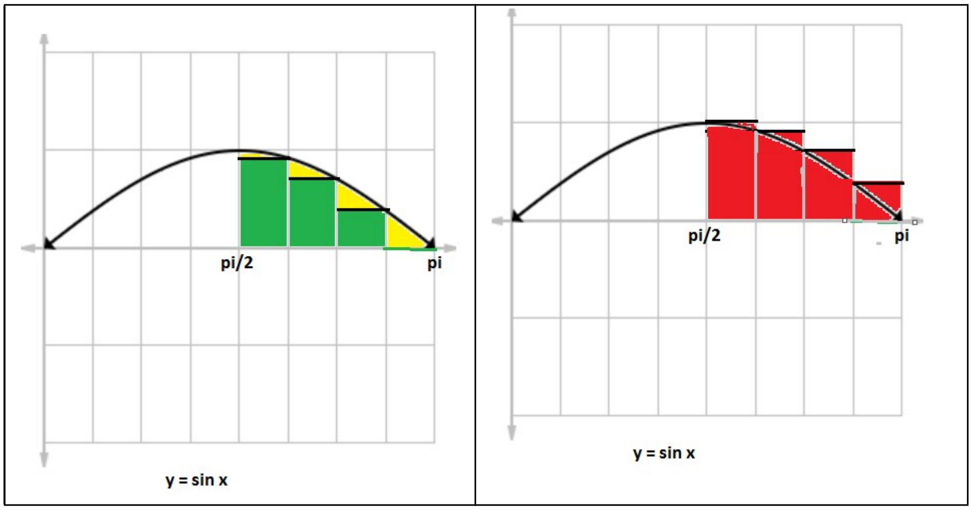
More completely,

$$R_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = \sum_{i=1}^{4} f(\tfrac{\pi}{2} + \tfrac{i\pi}{8}))\Delta x = \sum_{i=1}^{4} \sin(\tfrac{\pi}{2} + \tfrac{i\pi}{8}))(\tfrac{\pi}{8})$$

$$= \sin(\tfrac{5\pi}{8})\tfrac{\pi}{8} + \sin(\tfrac{6\pi}{8})\tfrac{\pi}{8} + \mathbf{\sin(\tfrac{7\pi}{8})\tfrac{\pi}{8}} + \sin(\tfrac{8\pi}{8})\tfrac{\pi}{8}$$

$$= \left(\sin(\tfrac{5\pi}{8}) + \sin(\tfrac{6\pi}{8}) + \sin(\tfrac{7\pi}{8}) + \sin(\tfrac{8\pi}{8})\right)\tfrac{\pi}{8} \approx 0.7908, \text{ and}$$

$$L_4 \quad = \left(\sin(\tfrac{4\pi}{8}) + \sin(\tfrac{5\pi}{8}) + \sin(\tfrac{6\pi}{8}) + \sin(\tfrac{7\pi}{8})\right)\tfrac{\pi}{8} \approx 1.1835$$



Since $f(x) = \sin x$ is decreasing on the interval $[\frac{\pi}{2}, \pi]$, R_n will always underestimate the area A (see green rectangles on the left for $n = 4$) and L_n (see red rectangles on the right) will always overestimate the area A.

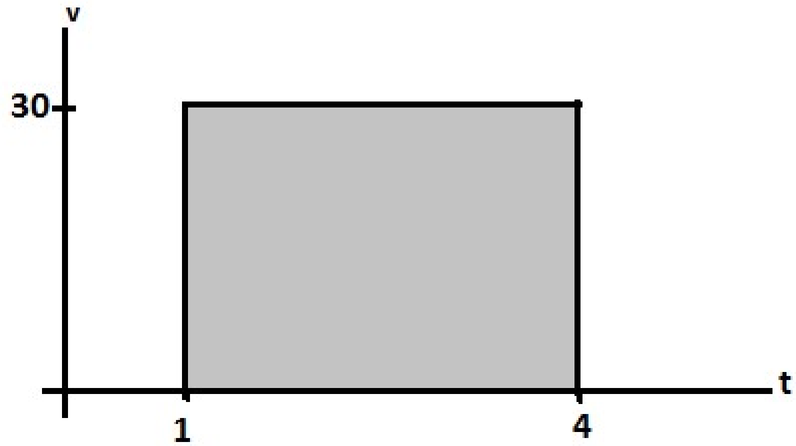
For some other values of n :

n	R_n	L_n
4	0.7908	1.1835
8	0.8986	1.0950
50	0.9842	1.0156
100	0.9921	1.0078
500	0.9984	1.0016
1000	0.9992	1.0008

which shows that $\qquad 0.9992 < A < 1.008$

The table might lead us to **guess** that the area A under $\sin x$ over the interval $[\frac{\pi}{2}, \pi] = 1$

Example: Suppose $v(t) = 30\,\text{m/sec}$ is the velocity (constant) of a point moving along a straight line at time t sec, where $1 \le t \le 4$:

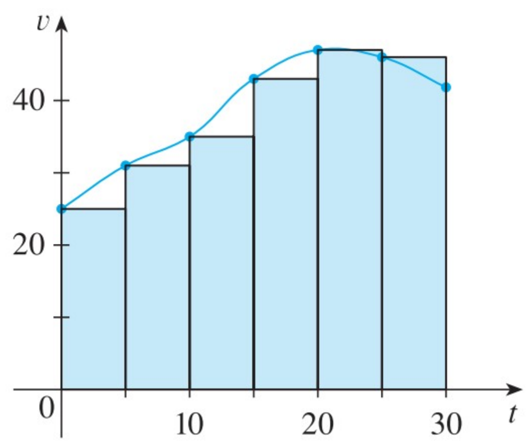


The area under the graph $\quad 30(3) = 90 \quad$ has a physical interpretation:

$$\text{(rate)(time)} = \text{distance}$$
$$(m/sec)(sec) \qquad (m)$$

That is, the area under this constant velocity function over the time interval $[1, 4]$
also represents distance traveled for $1 \le t \le 4 = 90$m

If the velocity function $v(t) \ge 0$ is not constant then we can approximate the area under the graph using a Riemann sum (L_6 in the picture below).



$$\text{Sum of blue rectangle areas} \approx \begin{cases} \text{area under the graph} \\ \text{total distance traveled for } 0 \le t \le 30 \end{cases}$$

A Riemann sum has both a geometric and a physical meaning here. Taking the limit (using L_n) :

$$\lim_{n \to \infty} L_n = \begin{cases} \text{area under the graph of } v(t) \text{ and above } [0, 30] \\ \text{total distance traveled for } 0 \le t \le 30 \end{cases}$$

We now restate our procedure and make a definition. The only real change is that we no longer require $f(x) \geq 0$.

Definition Suppose $f(x)$ continuous on $[a, b]$ (or even that $f(x)$ has a finite number of jump discontinuities)

(We no longer ask that $f(x) \geq 0$)

Divide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$..
Let $x_1^*,\ x_2^*, \ldots x_n^*$ be "sample points" chosen from each of the subintervals.

Form the Riemann sum $S = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$

$\lim_{n\to\infty} S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ is a number called the integral of $f(x)$ from a to b which is denoted by the symbol $\int_a^b f(x)\, dx$

Thus $$\int_a^b f(x)\, dx = \lim_{n\to\infty} S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

It can be proven that when f is continuous (or has a finite number of jump discontinuities) that this limit always exists and does not depend on how the x_i's are chosen. So we could also write, for example,

$$\int_a^b f(x)\, dx = \begin{cases} \lim_{n\to\infty} R_n & \text{or} \\ \lim_{n\to\infty} L_n \end{cases}$$

- The numbers a and b are called the "limits" for the integral.
- Think of $f(x)\, dx$ as a single symbol, not a multiplication: the x just serves as a reminder of what variable is involved in the integral which can be handy in complicated problems. However if the reminder isn't needed, someone might drop out the x's altogether and just write $\int_a^b f$.

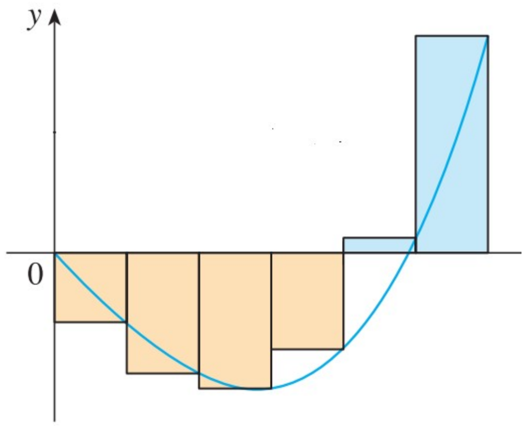
So, if $y = f(x) \geq 0$, $\int_a^b (f(x) dx =$ the area under the graph over $[a, b]$

if $v(t) \geq 0$ is a velocity function, $\int_a^b v(t) dt =$ the total distance traveled between times $t = a$ and $t = b$.

- the letter used for the variable isn't important. We could have $y = f(x) = x^2$ or velocity $v(t) = t^2$

$\int_0^1 x^2\, dx = \frac{1}{3}$ (*worked out in preceding lecture*) $=$ area under the graph
$\int_0^1 t^2\, dt = \frac{1}{3}$ (*total distance traveled).*

What happens if $f(x)$ is sometimes negative? For example:



(*In the picture, the rectangles are "right endpoint rectangles")*

In the Riemann sum corresponding to this picture:

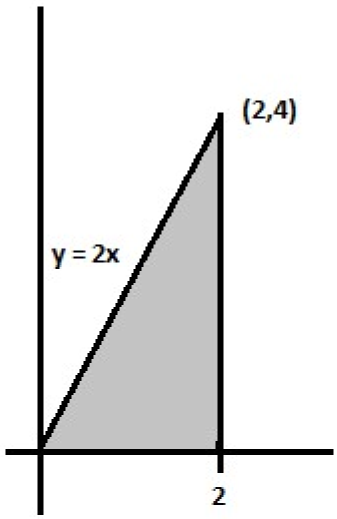
$$R_6 = \boldsymbol{f(x_1^*)\Delta x + \ldots + f(x_4^*)\Delta x} + \boldsymbol{f(x_5^*)\Delta x + f(x_6^*)\Delta x}$$

$f(x_1^*)$, ..., $f(x_4^*)$ are negative numbers, so each term like $f(x_1^*)\Delta x$ is the negative of the area of the corresponding rectangle in the picture; the blue terms represent positive areas, as before.

In each Riemann sum R_n there will "cancellation" of areas that count as positive and negative. The result is that the limit $\lim_{n\to\infty} \mathrm{R}_n = \int_0^3 f(x)\,dx =$ "net signed area" $=$ "area above the x-axis" $-$ "area below the $x-$ axis."

Example: Find $\int_0^2 2x\,dx$.

We know that the integral gives the shaded area, so by geometry, we know the answer:



Area under graph $= \frac{1}{2}(2)(4) = 4 = \int_0^2 2x\,dx$

But let's work out the integral using a limit of Riemann sums and, since it doesn't matter how we choose the sample points, we will use right endpoint Riemann sums R_n.

Q2: Which expression gives the right endpoint Riemann sum for $\int_0^2 2x\,dx$?

A) $R_n = \sum_{i=1}^{n} \frac{4i}{n} \cdot \frac{3}{n}$ B) $R_n = \sum_{i=1}^{n} \frac{4i}{n} \cdot \frac{2}{n}$ C) $R_n = \sum_{i=1}^{n} \frac{2i}{n} \cdot \frac{2}{n}$

D) $R_n = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{2}{n}$ E) $R_n = \sum_{i=1}^{n} \frac{4i}{n} \cdot \frac{1}{n}$

Answer: $\Delta x = \frac{2-0}{n} = \frac{2}{n}$*. The right endpoints of the* n *subintervals are* $\frac{2}{n},\ 2(\frac{2}{n}) = \frac{4}{n},\ 3(\frac{2}{n}) = \frac{6}{n}, ..., i(\frac{2}{n}), ..., n(\frac{2}{n}) = 2$

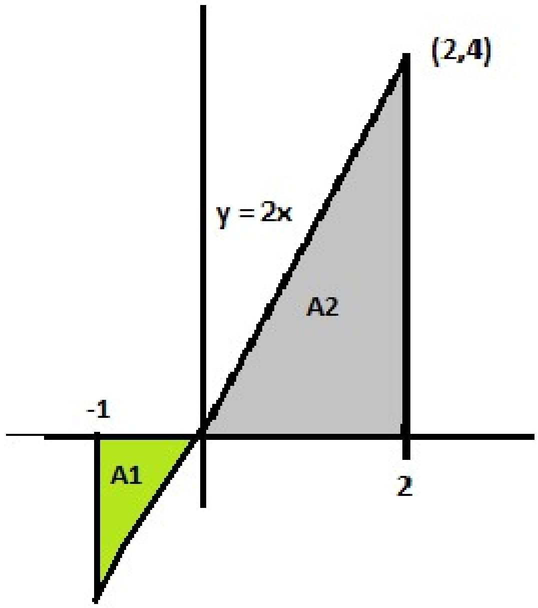
So $R_n = \sum_{i=1}^{n} f(i(\frac{2}{n}))\Delta x = \sum_{i=1}^{n}(2(i(\frac{2}{n}))\Delta x = \sum_{i=1}^{n}(\frac{4i}{n}) \cdot \frac{2}{n}$ *(Answer B)*

To continue:

$\int_0^2 2x\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i}{n^2} = \lim_{n\to\infty} \frac{8}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{8}{n^2}(\frac{n(n+1)}{2})$

$= \lim_{n\to\infty} \frac{4n^2+4n}{n^2} = 4$ *(the same answer, of course, as computed using the formula for the area of a right triangle).*

Example What is $\int_{-1}^{2} 2x\,dx$?



Just arguing geometrically $\int_{-1}^{2} 2x\,dx =$ area $A_2 -$ area $A_1 = 4 - 1 = 3$

Example (a hint of an easier way to do such calculations)

Suppose a point moves along a line with position $s = t^2 \quad 0 \le t \le 2)$
Then velocity is $v - \frac{ds}{dt} = 2t$

How far does it travel?

One solution : $s(2) - s(0) = 4 - 0 = 0$
(*since* $v \ge 0$*, there is no change in direction of motion, so this calculation gives total distance traveled*)

Another solution: distance traveled $= \int_0^2 v(t)\,dt = \int_0^2 2t\,dt = 4$ (*as calculated above*)

Of course, both methods produced the same solution. In other words

$\int_0^2 v(t)\,dt = s(2) - s(0)$

The integral, in this example, can be found by 1) finding an antiderivative $s = t^2$

for the function $v = 2t,$ substituting to get $S(2)$ and $s(0)$, then subtracting.

We <u>will see</u> that a method like this always works $-$ but we're not quite there yet!