Review of Definition of the integral

$f(x)$ and $g(x)$ continuous on $[a, b]$ (*actually, a finite number of jump discontinuities is also ok).*

Go back to the definition of $\int_a^b f(x)\,dx$: what follows here uses the same notation, not repeated here.

$$\Delta x = \tfrac{b-a}{n},\ x_i^*\text{'s} = \text{“sample points”} - \textit{one chose in each subinterval}$$

$$\lim_{n\to\infty}\left(f(x_1^*)\Delta x + f(x_1^*)\Delta x + \ldots + f(x_1^*)\Delta x\right) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_1^*)\Delta x = \int_a^b f(x)\,dx$$

$$\lim_{n\to\infty}\left(g(x_1^*)\Delta x + g(x_1^*)\Delta x + \ldots + g(x_1^*)\Delta x\right) = \lim_{n\to\infty}\sum_{i=1}^{n} g(x_1^*)\Delta x = \int_a^b g(x)\,dx$$

Adding (or subtracting) these limits and combining terms gives

$$\begin{aligned}\int_a^b f(x)\,dx + \int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_1^*)\Delta x \pm \lim_{n\to\infty}\sum_{i=1}^{n} g(x_1^*)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(x_1^*) \pm g(x_1^*)\right)\Delta x \\ &= \int_a^b \left(f(x) \pm g(x)\right)dx\end{aligned}$$

In other words:

- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

  “the integral of the sum of the integrals”

  “the integral of a difference is the difference of the integrals”

Multiplying by a constant c in the definition of $\int_a^b f(x)\,dx$ gives

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_1^*)\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_1^*)\Delta x = c\int_a^b f(x)\,dx$$

In other words:

- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

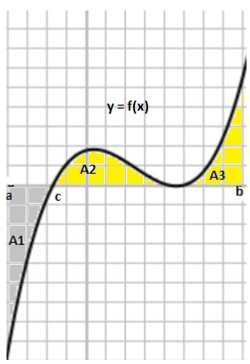
- $\int_a^a f(x)\,dx = 0$ (because if $a = b$ in the definition, then $\Delta x = 0$, so every Riemann sum is 0)

- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (because if a and b are reversed in the definition, then $\Delta x = \frac{a-b}{n} < 0$. So the sign of every Riemann sum is reversed and therefore the sign of the limit of the Riemann sums (the integral) is reversed.

- $\int_a^{\color{red}c} f(x)\,dx + \int_{\color{red}c}^b f(x)\,dx = \int_a^b f(x)dx$ (*)

For example, in the picture below this just says that

$-A_1 + (A_2 + A_3) = -A_1 + A_2 + A_3$



But it's important to note that the (*) property of integrals is true no matter what the order or size of $a,\ b,\ c$ – the only thing required is that the upper limit on the first integral match the lower limit on the second integral in (*) as highlighted in red, above.

For example: $\int_b^{\color{red}a} f(x)\,dx + \int_{\color{red}a}^c f(x)\,dx = \int_b^c f(x)\,dx$

To check this in the picture above:

LEFT SIDE:

$$\int_b^{\color{red}a} f(x)\,dx + \int_{\color{red}a}^c f(x)\,dx \quad = -\int_a^b f(x)\,dx \quad + \int_a^c f(x)\,dx$$

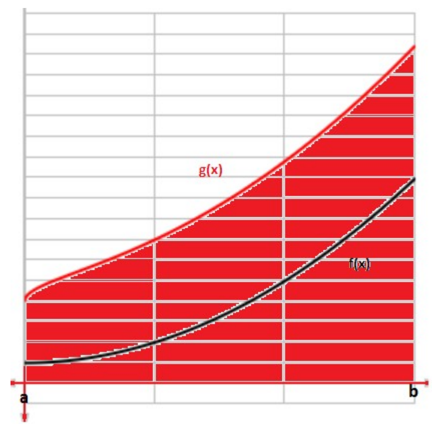
$$= -(-A_1 + A_2 + A_3) \quad + (-A_1) = -A_2 - A_3$$

RIGHT SIDE: $\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx = -(A_2 + A_3) = -A_2 - A_3$

- If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
  because, in the Riemann sums, every $f(x_i^*) \le g(x_i^*)$. Therefore
  (any Riemann sum for f) $\le$ (the corresponding Riemann sum for g)
  so, taking limits gives $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

When $f(x) \ge 0$ and $g(x) \ge 0$ this makes perfect sense interpreted as areas:

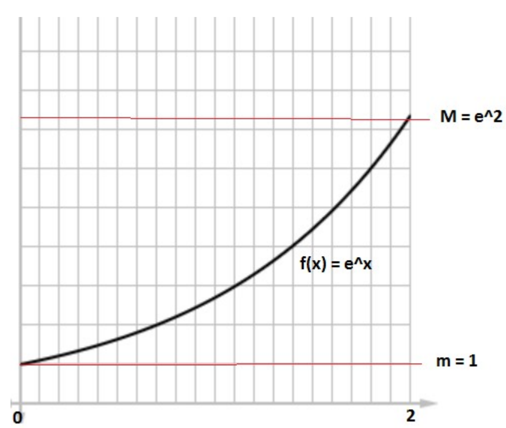
(area under the lower graph $f(x)$) $\le$ (area under the higher graph $g(x)$)



In particular, if

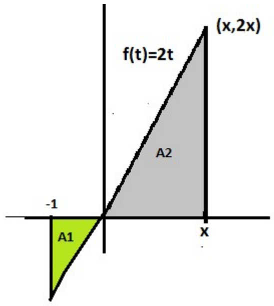
$$\begin{array}{rcccl} m & \le & f(x) & \le & M \text{ on } [a,b], \text{ then} \\ \int_a^b m\,dx & \le & \int_a^b f(x)\,dx & \le & \int_a^b M\,dx, \quad \text{that is,} \\ m(b-a) & \le & \int_a^b f(x)\,dx & \le & M(b-a) \end{array}$$

Example: $\int_0^2 e^x\,dx$



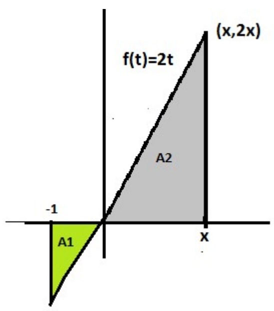
$$\begin{array}{lcl} 1 & \le e^x \le & e^2 \text{ on } [0,2] \\ \int_0^2 1\,dx = 2 & \le \int_0^2 e^x\,dx \le & \int_0^2 e^2\,dx = e^2(2-0) = e^2 \end{array}$$

If $x \geq 0,$ what is $\int_0^x 2t\,dt$?



The integral represents the area $A_2 = \frac{1}{2}x(2x) = x^2$ (by the formula for area of a triangle).

Q1 : What is $\int_0^{-1} 2t\,dt$?



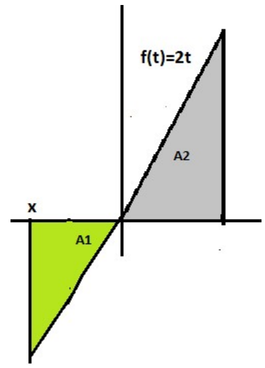
A) 0 B) -1 C) 1 D) 2 E) -2

Answer C: We can think of $\int_0^{-1} 2t\,dt$ *as a signed area, but that requires having the lower limit on the integral* (a) *be less than the upper limit* (b). *To make that true, we reverse the limits:*

$\int_0^{-1} 2t\,dt = -\int_{-1}^{0} 2t\,dt.$ *But* $\int_{-1}^{0} 2t\,dt$ *represents the* *negative of area* A_1

So $\int_0^{-1} 2t\,dt = -\int_{-1}^{0} 2t\,dt = -(-A_1) = A_1.$ *Since area* $A_1 = 1$ *(by triangle area formula), we get the final result* $\int_0^{-1} 2t\,dt = 1$

Q2: If $x < 0$, what is $\int_0^x 2t\,dt$?



A) x B) $-x$ C) $\frac{1}{3}x^3$ D) $-x^2$ E) x^2

Answer E) As in Q1, we can think of the integral as a (signed) area but that requires having the smaller number x as the lower limit for the integral. So we switch the limits to make that so:

$\int_0^x 2t\,dt = -(\int_x^0 2t\,dt)$. The integral $\int_x^0 2t\,dt$ gives the negative of area A_1, so
$\int_0^x 2t\,dt = -(\int_x^0 2t\,dt) = -(-A_1) = A_1$, and area $A_1 = \frac{1}{2}|x||2x|$
$= |x|^2 = x^2$

Define a new function $F(x) = \int_0^x 2t\,dt$. The "recipe" for $\int_0^x f(t)\,dt$ is very complicated, but since $f(t) = 2t$ is so simple we were able to work out by simple geometry that

$$F(x) = \int_0^x 2t\,dt = x^2 \text{ for all } x$$

For example, we know that $F(1) = \int_0^1 2t\,dt = 4$ and (above) that $F(-1) = 1$.

Notice (just an observation) that F is an antiderivative for f :

$$F'(x) = (x^2)' = 2x = f(x)$$

It turns out (next lecture) that part of the Fundamental Theorem of Calculus states that:

for any continuous function $f(t)$ we can define a new function by making use of an

integral:

$$F(x) = \int_0^x f(t)\,dt \text{ and that } F'(x) = f(x)$$

For example: $F(x) = \int_0^x e^{-t^2}dt$ (*there is no simpler formula for this function* $F(x)$!)
But $F'(x)$ is e^{-x^2} by the Fundamental Theorem of Calculus !