From previous lecture:

for $f(t) = 2t$: Define $F(x) = \int_0^x f(t)\,dt = \int_0^x 2t\,dt = x^2$

We worked out that $F(x) = \int_0^x 2t\,dt = x^2$ simplifies to x^2 using calculations of areas of triangles and properties of the integral.

Therefore we can compute $F'(x) = \frac{d}{dx}(x^2) = 2x$

What effect would it have to change the "0" in the integral to some other constant?

Q1: What is a simpler formula for $F(x) = \int_3^x 2t\,dt$?

(*Recall: you know* $\int_0^x 2t\,dt = x^2$*; think in terms of areas*)

A) x^2

B) $x^2 + 3x$

C) $x^2 - 3x$

D) $x^2 + 4$

E) $x^2 - 9$

Answer E: $\int_0^x 2t\,dt = \int_0^3 2t\,dt + \int_3^x 2t\,dt$ (*from previous lecture, and true whether or not* 3 *is between* 0 *and* x)

$\uparrow$

triangle area

$\downarrow$

$x^2 = 9 + \int_3^x 2t\,dt$

so $F(x) = \int_3^x 2t\,dt = x^2 - 9$. *Notice that it is* <u>*still*</u> *true that* $F'(x) = 2x$

A similar calculation will show you the $F(x) = \int_a^x 2t\,dt = x^2 + C$ where C is a constant (possibly negative). Therefore $F'(x) = 2x$ no matter how the constant a is chosen.

<u>In summary</u> For any choice of the constant a, $F(x) = \int_a^x f(t)\,dt = \int_a^x 2t\,dt$,
$F(x)$ is an antiderivative for $f(x)$: $F'(x) = 2x$.

Fundamental Theorem of Calculus (Part I)

If f is continuous on $[a, b]$ and we define $F(x) = \int_a^x f(t)\,dt$

then F is an antiderivative for f, that is, $f'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

We will talk some about why this is true in the next lecture.

Example:

If $F(x) = \int_3^x \underbrace{\sqrt{1+\cos^2 t}}_{f(t)}\,dt$. then $F'(x) = \frac{d}{dx}\int_3^x \sqrt{1+\cos^2 t}\,dt = \sqrt{1+\cos^2 x}$

Example: $F(x) = \int_x^3 \sqrt{1+\cos^2 t}\,dt$. The Fundamental Theorem (Part I) assume that the constant limit in the integral is the lower limit/ We need to put the problem into that form before we can apply the Fundamental Theorem:

$$F(x) = \int_x^3 \sqrt{1+\cos^2 t}\,dt = -\int_3^x \sqrt{1+\cos^2 t}\,dt, \text{ so}$$

$$F'(x) = -\sqrt{1+\cos^2 x}$$

Q2: Suppose $y = G(x) = \int_0^{\sin x} \sqrt{2+t}\,dt$. What is $\frac{dy}{dx} = G'(x)$?

(*Hint: Let $u = \sin x$ and use the chain rule:* $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$)

A) $G'(x) = \sqrt{2+\sin x}$
B) $G'(x) = \sqrt{2+\cos x}$
C) $G'(x) = (\sin x)\sqrt{2+\sin x}$
D) $G'(x) = (\cos x)\sqrt{2+\sin x}$
E) $G'(x) = (\cos x)\sqrt{2+x}$

We have $y = \int_0^u \sqrt{2+t}\,dt$, where $u = \sin x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. From the Fundament Theorem Part I, $\frac{dy}{du} = \sqrt{2+u}$

so $\quad G'(x) = \frac{dy}{dx} = \sqrt{2+u}\,\frac{du}{dx} = \sqrt{2+\sin x}\,(\cos x)$

Consider the function $G(x) = \int_0^{\sin x} \sqrt{2+\cos t}\, dt$. Let's create a rough graph.

Since $\sin(x+2\pi) = \sin x$, it must be true that $G(x+2\pi) = G(x)$, so $G(x)$ is periodic with period 2π. We only need to try to draw the graph over the interval $[0, 2\pi]$.

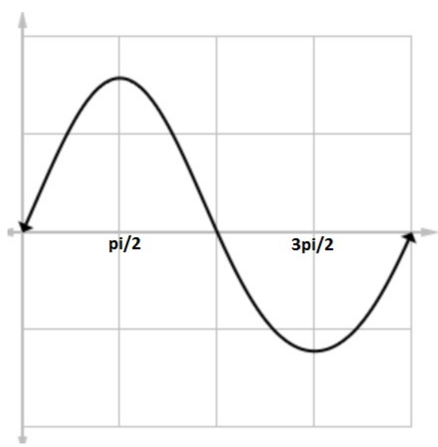
We can compute a few values of the function easily: $G(\pi) = \int_0^{\sin \pi} \sqrt{2+t}\ dt$ $= \int_0^0 \sqrt{2+t}\ dt = 0$.. Similarly, $G(0) = G(2\pi) = 0$

Since $\sqrt{2+\sin x} > 0$, the sign of $G'(x) = \sqrt{2+\sin x}\,(\cos x)$ is the same as the sign of cos :

$0 < x < \frac{\pi}{2}$	$\cos x > 0$	$G'(x) > 0$ so $G(x)$ is increasing
$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\cos x < 0$	$G'(x) < 0$ so $G(x)$ is decreasing
$\frac{3\pi}{2} < x < 2\pi$	$\cos x > 0$	$G'(x) > 0$ so $G(x)$ is increasing

So G has a local maximum at $x = \frac{\pi}{2}$ and a local minimum at $x = \frac{3\pi}{2}$.

We could, of course also compute $G''(x)$ to check concavity and inflection points, but won't do that here. The graph of $G(x)$ looks like



(Notice, from the graph, that it appears that $|G(\frac{\pi}{2})| > |G(\frac{3\pi}{2})|$ *)*

<u>Fundamental Theorem of Calculus (Part II)</u> Suppose f is continuous on $[a, b]$ and F is <u>any</u> antiderivative for f on $[a, b]$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Note: We often abbreviate the expression $F(b) - F(a) = F(x)\big|_a^b$

<u>Why?</u> $G(x) = \int_a^x f(t)\,dt$ is also an antiderivative for f on the interval $[a, b]$ so G and F must differ by a constant:

$G(x) = F(x) + C$. Evaluating at $x = a$. this gives
$G(a) = F(a) + C$. But $G(a) = \int_a^a f(t)\,dt = 0$. So
$0 \quad\;\; = F(a) + C$, so $C + -F(a)$.

Therefore

$G(x) = F(x) - F(a)$, and so

$$\begin{array}{l} G(b) = F(b) - F(a) \\ \| \\ \int_a^b f(t)\,dt \\ || \\ \int_a^b f(x)\,dx \end{array}$$

The Fundamental Theorem (Part II) lets us to a lot of computations easily.

Example: We worked out (using a limit of Riemann sums) that $\int_0^1 x^2 dx = \frac{1}{3}$.
This was a long calculation. Using the Fundamental Theorem (Part II), we could simply say:

Pick any antiderivative $F(x)$ for $f(x) = x^2$. For example, pick $F(x) = \frac{1}{3}x^3$.

Then $\int_0^1 x^2\,dx = F(x)|_0^1 = (\frac{1}{3}x^3)|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$.

Notice that any other antiderivative of x^2 would give the same answer. For example, if we had chosen $F(x) = \frac{1}{3}x^3 + 19$ then

$$\int_0^1 x^2\,dx = F(x)|_0^1 = (\tfrac{1}{3}x^3 + 19)|_0^1 = \tfrac{1}{3}(1 + 19)^3 - \tfrac{1}{3}(0 + 19)^3 = \tfrac{1}{3}$$

The constant "19" cancels out in the arithmetic.