The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Part I of the Fundamental Theorem is about differentiating a function defined by an integral. It says that g is an antiderivative for f.

Notice that in the formula:

The lower limit on the integral must be constant: a
The upper limit is the variable in the function $y = g(x)$
The derivative treats x as the variable: that is $g'(x)$ means $\frac{dy}{dx}$

So the conclusion reads $g'(x) = \frac{dy}{d\boldsymbol{x}} = \frac{d}{d\boldsymbol{x}} \int_a^{\boldsymbol{x}} f(t)\,dt = f(x)$

If the **red variables** didn't match (for example,

$\frac{dy}{d\boldsymbol{x}} = \frac{d}{d\boldsymbol{x}} \int_a^{\boldsymbol{u}} f(t)\,dt$, then the chain rule would

come into play:

$$\frac{dy}{d\boldsymbol{x}} = \frac{dy}{d\boldsymbol{u}}\frac{d\boldsymbol{u}}{d\boldsymbol{x}} = \left(\frac{d}{d\boldsymbol{u}} \int_a^{\boldsymbol{u}} f(t)\,dt\right)\frac{d\boldsymbol{u}}{d\boldsymbol{x}} = f(u)\frac{d\boldsymbol{u}}{d\boldsymbol{x}}$$

Example: if $g(x) = \int_0^x 2t\,dt$, then $g'(x) = 2x$

if $g(x) = \int_0^{x^2} 2t\,dt$, then (*letting* $u = x^2$)
$g'(x) = f(u)\frac{du}{dx} = 2(x^2)(2x) = 4x^3$

Q1: If $F(x) = \int_0^x (2 + \sin t)\,dt$, what is $F'(\frac{\pi}{2})$?

A) 0 B) 1 C) 2 D) 3 E) 4

Answer D: $F'(x) = 2 + sin\,x$, *so* $F'(\frac{\pi}{2}) = 2 + sin(\frac{\pi}{2}) = 2 + 1 = 3$

During the lecture also gave an infrmal argument about why the Fundamental Theorem, Part I, is true. This is also discussed in the textbook (informally, and then more formally) so it is not reproduced here again.

Question: What is $\frac{d}{dx}\int_0^7 (2+\sin t)\,dt$?

Answer: the upper limit, "7" is not a variable:

$$\int_0^7 (2+\sin t)\,dt = \text{a number (constant)}$$

so $\frac{d}{dx}\int_0^7 (2+\sin t)\,dt = 0$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

The second part of the Fundamental Theorem is more computational. Assuming f is continuous, it says that you can calculate the value of $\int_a^b f(t)\,dt$ by (somehow) finding any antiderivative for f : a function F for which $F'(x) = f(x)$.

Then $\int_a^b f(x)dx = F(b) - F(a)$ (which we abbreviate as $f(x)|_a^b$)

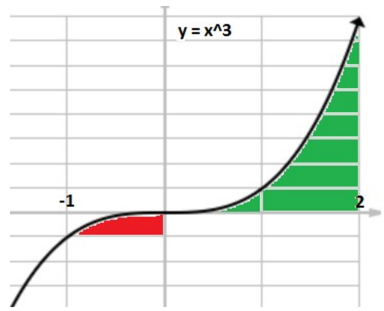
Finding antiderivatives is not as "mechanical" a process as finding derivatives. We will look at several useful tools for finding nice formulas more complicated antiderivatives in Calculus II. For now, about all we can do is be inventive using our differentiation formulas "backwards."

Just for example:

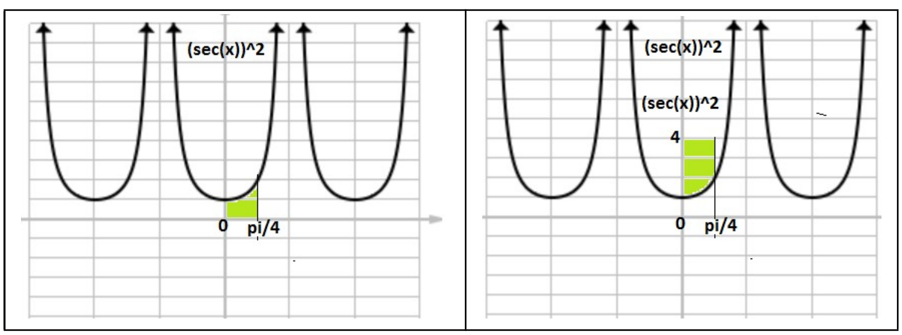
$f(x)$	$F(x)$	
x^3	$\frac{1}{4}x^4 + C$	(where C is any constant)
x^n	$\frac{1}{n+1}x^{n+1} + C$, provided $n \neq -1$	
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$	(or just $\ln x$ when a problem only involves $x > 0$)
$\sec^2 x$	$\tan x + C$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$	
$\ln x$	$x\ln x - x + C$	*Here, it's not obvious how you might <u>discover</u> that $x\ln x - x + C$ is an antiderivative for $\ln x$, but it is easy to <u>check</u> that it works:* $\frac{d}{dx}(x\ln x - x + C) - x(\frac{1}{x}) + (\ln x)(1) - 1$ $= \ln x$

Example: $\int_{-1}^{2} x^3 dx = (\frac{1}{4}x^4)|_{-1}^{2} = (\frac{1}{4}2^4) - (\frac{1}{4}((-1)^4) = \frac{15}{4}$

Notice that an "area interpretation" of this integral in the picture below is "Green Area – "Red Area"



Q2: What is the shaded area in the picture on the left? Example



A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{\pi}{2}$ D) 1 E) $\frac{3\pi}{2}$

Answer D: In the left picture, green area $= \int_0^{\pi/4} sec^2\, x\, dx = tan\, x|_0^{\pi/4} =$ $tan(\frac{\pi}{4}) - tan(0) = 1 - 0 = 1$

3: What is the shaded area in the picture (above) on the right?

A) π B) $\pi - 1$ C) $\frac{\pi}{2} + 1$ D) 2π E) 2

$y =$ *Answer B: We can compute*

Area under the line $y = 4$ *but above the graph of* $y = sec^2\, x$ *over the interval* $[0, \frac{\pi}{4}]$

$=$ *(Rectangle area under the line* $y = 4$ *and above* $[0, \frac{\pi}{4}]) - ($*Area in Question 2*$)$

$= \frac{\pi}{4}(4) - 1 = \pi - 1.$

In general, if $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, it's easy to see a formula to try to find the area between the tow graphs easily

(Green) area under $f(x)$ and above $g(x)$ between a and b

$= $ (area under $f(x)$ and above x-axis)

$-$ (area under $g(x)$ and above x-axis)

$= \int_a^b f(x)\,dx \quad - \int_a^b g(x)\,dx$

$= \int_a^b (f(x) - g(x))\,dx$

($= \int_a^b$ "top boundary curve $-$ bottom boundary curve" dx

In the preceding example, the area between $y = 4$ and $y = \sec^2 x$ between 0 and $\frac{\pi}{4}$ could be written as $\int_0^{\pi/4} (4 - \sec^2 x)\,dx = \int_0^{\pi/4} 4\,dx - \int_0^{\pi/4} \sec^2 x\,dx = \pi - 1.$

A side comment for those interested: if we want to evaluate, for example, $\int_1^5 \sqrt{1+x^2}\,dx$, we need an antiderivative $F(x)$ for $f(x) = \sqrt{1+x^2}$ The Fundamental Theorem, Part I, automatically gives us an antiderivative (one with a fancy definition):

$F(x) = \int_1^x \sqrt{1+t^2}\,dt$. Could we use this antiderivative F to evalute our integral? The theory would work out perfectly, but it wouldn't help us computationally. Watch what would happen:

$\int_1^5 \sqrt{1+x^2}\,dx = F(x)|_1^5 = F(5) - F(1)$

$= \int_1^5 \sqrt{1+t^2}\,dt - \int_1^1 \sqrt{1+t^2}\,dt = \int_1^5 \sqrt{1+t^2}\,dt - 0 = \int_1^5 \sqrt{1+t^2}\,dt$

which is true but still doesn't help us find a numeric value for $\int_1^5 \sqrt{1+x^2}\,dx$!!

Another way of looking at the Fundamental Theorem, Part II:

Suppose $y = F(x)$ and $\frac{dy}{dx} = \frac{dF}{dx} = F'(x) = f(x)$

alternate notations for $F'(x)$

so $\int_a^b f(x)dx = F(b) - F(a)$

$$\int_a^b f(x)dx = F(b) - F(a)$$
$$||$$
$$\int_a^b F'(x)dx$$
$$||$$
$$\int_a^b \frac{dF}{dx}(x)dx$$

The integrals on the left are all the same, just using different notation for $f(x)$. The notation $\frac{dF}{dx}$ simply reminds us that the $F'(x)$ gives the rate of change of F with respect to the variable x.

So the Fundamental Theorem, Part II says

$$\int_a^b \frac{dF}{dx}(x)dx = F(b) - F(a), \quad \text{or}$$

(*) $\int_a^b (\text{rate of change of } F) = F(b) - F(a) =$ net change in F.

The textbook calls (*) the Net Change Theorem

Some related examples:

Example

$s = f(t) = 4t - t^2 =$ position (m) of a point moving along a straight line

$\frac{ds}{dt} = v(t) =$ velocity $= 4 - 2t$

Notice these positions: $s(0) = 0, \quad s(2) = 4, \quad s(4) = 0$

(the point is moving right of $0 \leq t \leq 2$*, and left for* $2 \leq t \leq 4$ *– as you can see by checking the sign of the velocity* $v(t)$*)*

$\int_0^2 v(t)\,dt = \int_0^2 \frac{ds}{dt}\,dt = \int_0^2 (\text{rate of change of } s)$

$||$

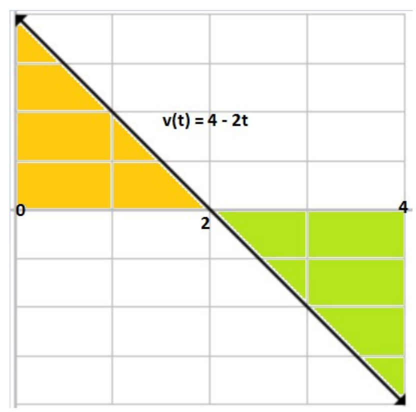
$\int_0^2 4 - 2t = s(t)|_0^2 = s(2) - s(0)$ (net change in s between $t = 0$ and $t = 2$)

$$= 4 - 0 = 4 \quad \text{(m)}$$

Notice also that $\int_0^4 v(t)\,dt = \int_0^4 4 - 2t\,dt = s(t)|_0^4 = 0 - 0 = 0$

Beween times 0 and 4 the point moves right to 2, and then back to 0 : at $t = 4$, it has returned to its starting position $-$ so the <u>net change in position</u> in 0! (The *<u>total distance traveled</u>, ignoring postive and negative directions is* 4 : *2 m to the right followed by* 2 *m to the left.)*

If you think of $\int_0^4 4 - 2t\,dt$ *in terms of areas, you can see that the the areas above and below the* t *axis are equal and cancel out when the integral is computed.*



Don't confuse <u>net change in position</u> with <u>total distance travelled</u>: they are the same <u>IF</u> the point is always moving to the right during the time interval (as in the case, here, for $0 \leq t \leq 2$) but are different when the point moves both left and right during the time interval.

If you wanted to find <u>total distance traveled</u> you would want to ignore the sign of $v(t)$: in other words, total distnace traveled $= \int_a^b |v(t)|\,dt = \int_a^b (\text{speed})\,dt$. To actually compute,for example, $\int_0^4 |v(t)|dt = \int_0^4 |4 - 2t|\,dt$, you would need to remove the abolsute value signs by figuring out that

for $0 \leq t \leq 2$	$4 - 2t > 0$	so	$\|4 - 2t\| = 4 - 2t$
for $2 \leq t \leq 4$	$4 - 2t < 0$	so	$\|4 - 2t\| = 2t - 4$

Then $\int_0^4 |4 - 2t|\,dt = \int_0^2 (4 - 2t)dt + \int_2^4 (2t - 4)dt = 4 + 4 = 8$ m total distance traveled.

Example Let $V(t)$ = the volume of water (ft^3) in a tank at time t (min)

Water is added to the tank at a rate of $1 + e^{-t}$ ft^3, so $\frac{dV}{dt} = 1 + e^{-t}$

Then $\int_0^3 \frac{dV}{dt}\, dt = V(3) - V(0)$, that is

$\int_0^3$ (rate of change of V) = net change in V bfrom $t = 0$ to $t = 3$

= amount of water added from $t = 0$ to $t = 3$

Specifically, $\int_0^3 1 + e^{-t}\, dt = (t - e^{-t})|_0^3 = (3 - e^{-3}) - (0 - e^0) = 4 - \frac{1}{e^3}$ (ft^3)