The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

If u is a function of x, then the chain rule is involved:

$$y = \int_a^u f(t)\,dt \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}\int_a^u f(t)\,dt\,\frac{du}{dx} = g(u)\frac{du}{dx}$$

Q1 If $G(x) = \int_0^{x^2} \frac{2}{1+t^2}\,dt$, what is $G'(1)$?

A) 0 B) 1 C) 2 D) 3 E) 4

Answer C: $G'(x) = \frac{d}{dx}\int_0^u \frac{2}{++t^2}\,dt$ (where $u = x^2$)

$$= \frac{d}{du}\int_0^u \frac{2}{1+t^2}\,dt \cdot \frac{du}{dx} = \frac{2}{1+u^2}(2x) = \frac{4x}{1+x^4}$$

so $g'(1) = \frac{4}{2} = 2$

Example Find $G'(x)$ if $G(x) = \int_x^{x^2} \frac{2}{1+t^2}dt$.

Since the variable x appears in both the upper and lower limits of the integral, the Fundamental Theorem doesn't apply directly. Break the integral into 2 pieces:

$$G(x) = \int_x^0 \frac{2}{1+t^2}\,dt + \int_0^{x^2} \frac{2}{1+t^2}\,dt.$$

Since the first integral has the constant limit, 0 as the upper limit, we need to reverse the limits in order to apply the Fundamental Theorem to it. So

$$\begin{aligned} G(x) &= -\int_0^x \frac{2}{1+t^2}\,dt + \int_0^{x^2} \frac{2}{1+t^2}\,dt. \\ &= -\frac{2}{1+x^2} + \frac{4x}{1+x^4} \end{aligned}$$

“Breaking” the integral into two parts <u>at 0</u> was a arbitrary choice: it would have led to the same answer to write $G(x) = \int_x^{17} \frac{2}{1+t^2}\,dt + \int_{17}^{x^2} \frac{2}{1+t^2}\,dt$ (*go through the steps above*!)

However, in certain cases “breaking the integral into two integrals” requires a little care:

for example , we <u>can't</u> write $\int_1^x \frac{1}{t} dt = \int_1^{-1} \frac{1}{t} dt + \int_{-1}^x \frac{1}{t} dt$

The integral $\int_1^{-1} \frac{1}{t} dt$ doesn't make sense since the function $\frac{1}{t}$ has a vertical asymptote at $t = 0$ (*draw the graph of* $\frac{1}{t}$!). We only defined the integral $\int_a^b f(t)dt$ when f is continuous (or, at worst, has just a finite number of jump discontinuities) in the interval $[a, b]$)

There's no problem, above, with wherever you choose to “break the integral into two pieces.” For example, no matter what c is chosen

$$\int_x^{x^2} \frac{2}{1+t^2} dt = \int_x^c \frac{2}{1+t^2}\,dt + \int_c^{x^2} \frac{2}{1+t^2}\,dt \text{ is OK}$$

since $\frac{2}{1+t^2}$ is guaranteed to be continuous on both $[x, c]$ and $[c, x^2]$.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Since the Fundamental Theorem part II requires finding an antiderivative for $f(x)$, a new notation is handy:

"the antiderivatives for $f(x)$ are $F(x) + C$" means the same thing as

$\int f(x)\,dx = F(x) + C$

For example: $\int x^4 dx = \frac{1}{5}x^5 + C$ (a bunch of functions: all the antiderivatives of x^4)

$\int_0^1 x^4 dx = \frac{1}{5}x^5|_0^1 = \frac{1}{5} - 0 = \frac{1}{5}$

$\int_0^1 x^4 dx$ is a <u>number</u>

$\int x^4 dx$ is a <u>bunch of functions</u>.

$\int_0^1 x^4 dx$ is called a <u>definite integral</u>

$\int x^4 dx$ is called an <u>indefinite integral</u> (which means the same as <u>the antiderivatives for x^4</u>)

<u>Example</u> We don't have many tools for finding antiderivatives yet: just remembering the differentiation formulas and thinking backwards. Sometimes some algebraic manipulation lets you recognize what to do.

$$\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$$

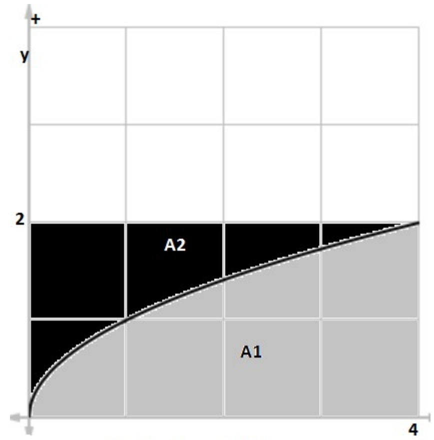
To find the integral, we need an antiderivative for $\frac{1+\cos^2\theta}{\cos^2\theta}$. We can find one with a little manipulation:

$$\int \frac{1+\cos^2\theta}{\cos^2\theta}\,dx = \int \frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\,d\theta = \int \sec^2\theta\ + 1\,d\theta = \tan\theta + \theta + C$$

so, using the Fundamental Theorem (Part II)

$$\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = (\tan\theta + \theta)|_0^{\pi/4} = (\tan\frac{\pi}{4} + \frac{\pi}{4}) - (\tan 0 + 0) = 1 + \frac{\pi}{4}.$$

Q2 Find the areas A1 and A2:



Graph of y = sqrt(x)

A) Area A1 $= 4$, Area A2 $= 4$

B) Area A1 $= 5$, Area A2 $= 3$

C) Area A1 $= \frac{25}{4}$, Area A2 $= \frac{7}{4}$

D) Area A1 $= \frac{23}{8}$, Area A2 $= \frac{9}{8}$

E) Area A1 $= \frac{16}{3}$, Area A2 $= \frac{8}{3}$

Answer E: Area $A1 = \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \frac{2}{3}x^{\frac{3}{2}}|_0^4 = \frac{2}{3}(8) - \frac{2}{3}(0) = \frac{16}{3}$

$$\textit{Area } A2 = \int_0^4 (2 - \sqrt{x}\,)dx = \int_0^4 2\,dx - \int_0^4 \sqrt{x}\,dx$$

$$= \textit{(rectangle area)} - A1$$

$$= 8 - \frac{16}{3} = \frac{8}{3}.$$

<u>Notice</u>: *the curve is* $y = \sqrt{x}$ $(0 \le x \le 4$ *which we could rewrite as*
$x = y^2$ $(0 \le y \le 2)$

Rotating your head clockwise by 90°. you can think of area $A2$ *as the area "under" the graph of* $x = y^2$ *and "above" the interval* $0 \le y \le 2$. *So* $A2 = \int_0^2 y^2\,dy = \frac{y^3}{3}|_0^2 = \frac{8}{3}$.

And, with this sideways turn of the head, $A! = \int_0^2 (4 - y^2)\,dy = (4y - \frac{y^3}{3})|_0^2 = \frac{16}{3}$.

Suppose $y = F(x)$ and $\frac{dy}{dx} = \frac{dF}{dx} = F'(x) = f(x)$

alternate notations for $F'(x)$

so $\int_a^b f(x)dx = F(b) - F(a)$ (Fundamental Theorem, Part II)

$$\int_a^b f(x)dx \\ || \\ \int_a^b F'(x)dx \\ || \\ \int_a^b \frac{dF}{dx}(x)dx$$

The integrals on the left are all the same, just using different notation for $f(x)$. But notation $\frac{dF}{dx}$ <u>reminds</u> us that the $F'(x)$ gives the <u>rate of change of F with respect to the variable x.</u>

So the Fundamental Theorem, Part II can be restated as

Net Change Theorem

$$\int_a^b \frac{dF}{dx}(x)dx = F(b) - F(a), \quad \text{or}$$

(*) $\int_a^b (\text{rate of change of } F) = F(b) - F(a) = \text{net change in } F.$

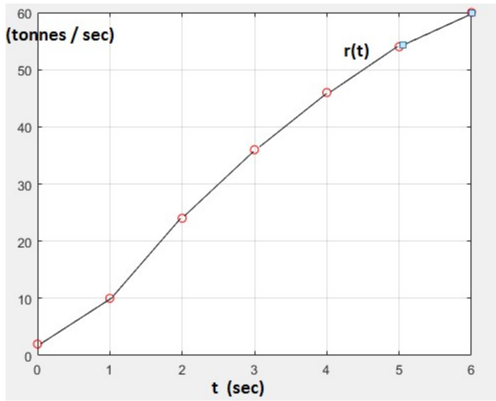
<u>Example</u> A volcano erupts at time $t = 0$. $Q(t)$ is the total number of tonnes of material ejected into the atmosphere by time t, and so $\frac{dQ}{dt}$ = the rate at which material is ejected into the atmosphere at time t (tonnes/sec).

(*A tonne is a "metric ton" = 1000 kg*)

Some collected data is arranged in a table

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

We can plot the data to see an <u>approximate</u> graph for $r(t)$, created by joining the data points from the table with straight line segments.



How much material is ejected into the atmosphere by time $t = 6$?

Use the Net Change Theorem:

$$\int_0^6 r(t)\,dt = \underset{\substack{\uparrow \\ \text{rate of change of } Q}}{\int_0^6 \frac{dQ}{dt}\,dt} = \underset{\substack{\uparrow \\ \text{net change in } Q}}{Q(6) - Q(0)} = Q(6) \quad (\textit{since } Q(0) \text{ is } 0)$$

We can only estimate $\int_0^6 r(t)\,dt$ from the data (*since we don't have a formula for* $r(t)$)

We could do this several ways, getting a slightly different estimate each time. Based on the data its not completely clear which estimate is "better" but they are all in the same "ball park."

a) We can compute the <u>left endpoint Riemann sum</u> L_6 to estimate $\int_0^6 r(t)\,dt$:

Divide $[0,6]$ into 6 equal subintervals: $\Delta t = 1$, left endpoints are $0, 1, 2, 3, 4, 5$

$$\int_0^6 r(t)\,dt \approx L_6 \quad = r(0)\Delta t + r(1)\Delta t + r(2)\Delta t + r(3)\Delta t + r(4)\Delta t + r(5)\Delta t$$

$$= (r(0) + r(1) + r(2) + r(3) + r(4) + r(5))\Delta t = 172 \text{ (tonnes)}$$

b) Instead, we can compute the <u>right endpoint Riemann sum</u> R_6 :

$$\int_0^6 r(t)\,dt \approx R_6 \quad = r(0)\Delta t + r(1)\Delta t + r(2)\Delta t + r(3)\Delta t + r(4)\Delta t + r(5)\Delta t$$

$$= (r(1) + r(2) + r(3) + r(4) + r(5) + r(6))\Delta t = 230 \text{ (tonnes)}$$

c) We can <u>average these two estimates</u>: $\int_0^6 r(t)\,dt = \frac{L_6 + R_6}{2} = 201$ (tonnes).

Usually the average on L_n and R_n gives a better estimate for the integral than either of the separate estimates.

Extra Information: what does this average mean geometrically?

$(L_n + R_n)/2 =$

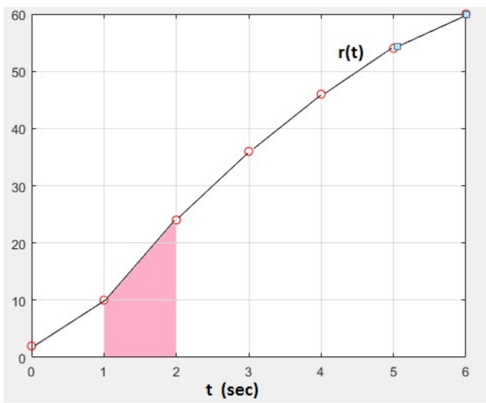
$$\Big((r(0) + r(1) + r(2) + r(3) + r(4) + r(5))\Delta t$$
$$+ \quad ((r(1) + r(2) + r(3) + r(4) + r(5) + r(6))\Delta t\Big)/2$$

$= (since\ (\Delta t = 1)\ \ \frac{r(0)+r(1)}{2} + \frac{\boldsymbol{r(1)+r(2)}}{\mathbf{2}} + \ldots + \frac{r(5)+r(6)}{2}$

$=$ a sum of trapezoidal areas.

For example, in the picture below,

$\frac{r(1)+r(2)}{2}$ is the area of the shaded trapezoid.



The sum of the trapezoidal areas approximates the area under the graph.
In general, the average $\frac{L_n+R_n}{2} = T_n$ is called the trapezoidal approximation to the integral. It's usually a better approximation to the actual value of an integral than either of L_n or R_n.

In this particular example, the pictured graph consists of straight line segments, and therefore T_6 *= sum of trapezoidal areas = the exact area under the pictured graph. But remember that the pictured graph is itself only an approximation (based on the tabular data) for the true function* $r(t)$. *Therefore* T_6 *is still only an approximation to the "actual" value of* $\int_0^6 r(t)\,dt$.