Here is a more careful discussion about why the Mean Value Theorem is true, based on things we've learned about calculus (rather than just an explanation by picture, as given above).

Suppose $f(x)$ is continuous on the closed interval $[a, b]$
and that $f'(x)$ exists at all points between a and b.

Let $g(x) =$ the linear function whose graph is the line through points A and B
This line has slope $\frac{f(b)-f(a)}{b-a} = m$, so $g'(x) = m$ (for every x)

We want to use an argument from calculus, not just an appeal to the picture, to explain why:

(*) there must be at least one point z between a and b where $f'(z) = m$.
In other words, there must be at least one z where the tangent line at $(z, f(z))$ is parallel to the straight line through A and B.

Notice that if $f(x)$ were to be a linear function, then the graph of $y = f(x)$ itself would be the straight line through A and B: in other words, we would have $f(x) = g(x)$. In that case, all z's would work: we'd have $f'(z) = g'(z)$ for every z and our explanation for the statement (*) would be finished!

Therefore we only need to explain why (*) is true when $f(x)$ is not a linear function, that is, when $f \neq g$.

Let $h(x)$ represent the vertical distance (in red, online) between $f(x)$ and $g(x)$.
Here are a few observations about this new function $h(x)$:

$h(x) = f(x) - g(x)$ for x in $[a, b]$

Since $f(a) = g(a)$ and $f(b) = g(b)$,
then (**) $h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$

$h(x)$ is continuous because f and g are continuous
so $h(x)$ has an absolute max value and an absolute min value on $[a, b]$ by the Extreme Value Theorem

(***) $h'(x)$ exists for all x in (a, b) because the same is true for f and g

What are the candidates for where the absolute max and min of h occur?

Candidates: a, b (the endpoints of $[a, b]$) and critical numbers of h in (a, b). These are the x's where $h'(x) = 0$ (by remark (***), there are no x's where $h'(x)$ does not exist!)

Could we have the absolute max value at one endpoint and the absolute min value at the other? That would mean

absolute <u>max value</u> $=$ absolute <u>min value</u> $= h(a) = h(b) = 0$ (*see comment* (***))
That would mean $f(x) - g(x) = h(x) = 0$ for all x, and therefore
$f(x) = g(x) =$ a linear function. But that's not true! Our discussion at this point is assuming that $f(x)$ <u>not</u> a linear function.

So it cannot be that <u>both</u> the absolute max value and absolute min values occur at an endpoint: at least one of them must occur at some other point (call it z) in (a, b). That z must be a critical point of h – a point where $h'(z) = 0$.

<u>For this z</u>, $h'(z) = f'(z) - g'(z) = 0$, so $f'(z) = g'(z) = m = \frac{f(b) - f(a)}{b - a}$.

This discussion proves

The Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable for all x between a and b, then there must be <u>at least one</u> z in (a, b) where

$$f'(z) = \frac{f(b) - f(a)}{b - a} \text{ or, equivalently, where}$$

$$f'(z)(b - a) = (f(b) - f(a))$$