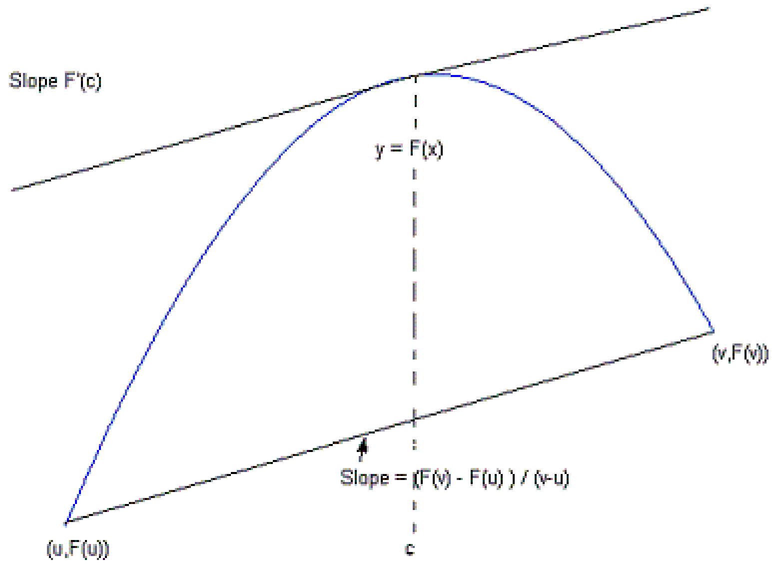
Why is the Evaluation Theorem (Fundamental Theorem of Calculus, Part II) True?

First, we want to recall the Mean Value Theorem (MVT) :

It states that if $F(x)$ is differentiable on an interval $[u, v]$, then there must be a point c between u and v where $\frac{F(v) - F(u)}{v - u} = F'(c)$, which we can rewrite as $F(v) - F(u) = F'(c)(v - u)$



We will use the Mean Value Theorem to see why the Evaluation Theorem is true.

<u>The Evaluation Theorem</u> says

If $F(x)$ is an antiderivative for $f(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>Why?</u> Here's the "recipe" for $\int_a^b f(x)\,dx$. <u>For a given value of n</u>, construct a Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ as follows: <u>subdivide</u> $[a, b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$, <u>choose sample points</u> x_i^* in each subinterval (<u>how</u> we choose is irrelevant: x_i^* could be a left endpoint, a right endpoint, a midpoint, or <u>any</u> other point in the subinterval), and <u>form the Riemann sum</u> $\sum_{i=1}^{n} f(x_i^*)\Delta x$. Then <u>take the limit of the Riemann sums</u> as $n \to \infty$ to get : $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$.

We will follow this recipe and use our freedom to choose the x_i^*'s in a very clever way. (*In fact, we'll let the Mean Value Theorem do the choosing for us!*)

So, for each value of n: the subintervals are $[a = x_0, x_1]$, $[x_1, x_2]$, ... , $[x_{i-1}, x_i]$, ..., $[x_{n-2}, x_{n-1}]$, $[x_{n-1}, x_n]$. Use the Mean Value Theorem on each subinterval as follows:

MVT says that there is a point c_i in the interval $[x_{i-1}, x_i]$ where $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x$.

We use this c_i to be our sample point x_i^* in $[x_{i-1}, x_i]$, and form the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} f(c_i)\Delta x$. We'll call the Riemann sum created in this way S_n.

Then take the limit as $n \to \infty$. We get that

$$\int_a^b f(x)dx = \lim_{n\to\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\to\infty} S_n$$

But we can greatly simplify the S_n's we chose : $S_n =$

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} f(c_i)\Delta x \qquad = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

$$= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + (F(x_3) - F(x_2)) + (F(x_4) - F(x_3))$$

$$+ \ldots + (F(x_{n-1}) - F(x_{n-2})) + (F(x_n) - F(x_{n-1}))$$

(*almost everything cancels out!*)

$$= F(x_n) - F(x_)) = F(b) - F(a)$$

So every $S_n = F(b) - F(a)$ (*because of the very clever way we chose the sample points* $x_i^* = c_i$). Therefore

$$\int_a^b f(t)\,dt \qquad = \lim_{n\to\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \qquad \lim_{n\to\to\infty} S_n$$

$$= \lim_{n\to\infty} (F(b) - F(a))$$

$$= \qquad F(b) - F(a),$$ which is what we wanted to show.