Name _____ **WU ID Number** _____

No calculators or notes are allowed on any part of this exam. As usual, "DNE" stands for "does not exist."

This exam should have 9 white pages (Part I), and 5 colored pages (Part II). The pages in each part are numbered in the upper right corner.

Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false questions (worth 1 point each), for a total of 75 points. Mark the correct answer on the answer card. You may use the test booklet for any work you want to write down, but for Part I, only the answer marked on the answer card will be graded.

Part II (separate colored pages) consists of 3 hand-graded questions, worth a total of 25 points. Also put your name on Part II, write your solutions on the colored sheets. Your written work will be scored by the instructor and TAs.

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1. A function $f(x)$ is defined for all values of x, but the only information that you know are the values shown in the table below:

x	y
0	1
0.001	0
0.01	1
0.1	2
1	3
2	4

Based on this information, what is your best estimate for the slope of the tangent line to the graph of $y = f(x)$ at the point $(0, 1)$?

A) -2500 B) -2000 C) -1500 **D) -1000**   E) -100

F) -5 G) 0   H) 20   I) 40   J) 2000

Among the points on the graph given in the table, the closest to $(0, 1)$ is $(0.001, 0)$, so we use the slope of "secant line" joining those two points to estimate the slope of the tangent line at $(0, 1)$:   slope of secant line $= \frac{f(0.001) - f(0)}{0.001 - 0} = \frac{0 - 1}{0.001} = -\frac{1}{\frac{1}{1000}} = -1000$.

2. Suppose $f(x) = \begin{cases} x^2 - 12 & \text{for } x < 4 \\ g(x) & \text{for } x \geq 4 \end{cases}$

If $f(x)$ is continuous at 4, what is $g(4)$?

A) -7 B) -5 C) -3 D) -1 E) 0

F) 2   **G) 4**   H) 6   I) 8   J) not enough information to determine an answer

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If $f(x)$ is continuous at 4, then $\lim_{x \to 4} f(x)$ exists <u>and</u> $\lim_{x \to 4} f(x) = f(4) = g(4)$.

Since we know that $\lim_{x \to 4} f(x)$ exists,

$$\lim_{x \to 4} f(x) = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (x^2 - 12) = 16 - 12 = 4.$$

So $\lim_{x \to 4} f(x) = 4 = g(4)$

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3. Suppose $f(x) = \begin{cases} x^2 + 5a & \text{when } x > 0 \\ (a + \frac{x^2+1}{2x^2+1}) \frac{\sin(2x)}{(2x)} & \text{when } x < 0 \end{cases}$, where a is a constant.

If $\lim_{x \to 0} f(x)$ exists, what is the value of a?

A) -2 B) -1 C) $-\frac{1}{8}$ D) $-\frac{1}{2}$ E) 0

F) $\frac{1}{2}$ **G) $\frac{1}{4}$**   H) $\frac{1}{8}$ I) 1   J) 2

For $\lim_{x \to 0} f(x)$ to exist, the limits at 0 from the left and right must both exist and be equal.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + 5a = 5a \quad \text{and} \quad \lim_{x \to 0+} f(x) = \lim_{x \to 0+} (a + \frac{x^2+1}{2x^2+1}) \frac{\sin(2x)}{(2x)}$$

$$= (a + 1) \lim_{x \to 0+} \frac{\sin(2x)}{(2x)} = (a+1)(1) = a + 1$$

So $5a = a + 1$ and therefore $a = \frac{1}{4}$.

4. The line $y = c$ is a horizontal asymptote for $y = f(x) = \dfrac{5(x-1)(2x-7)}{12(x-1)(x-4)}$.
What is c?

A) 5   B) $\frac{5}{12}$ C) $\frac{7}{4}$ D) $\frac{7}{12}$ **E) $\frac{5}{6}$**

F) 1   G) 0   H) $-\frac{5}{12}$ I) $\frac{5}{4}$ J) 2

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{5(x-1)(2x-7)}{12(x-1)(x-4)} = \lim_{x\to\infty} \frac{10x^2 - 45x + 35}{12x^2 - 60x + 48}$$

$$= \lim_{x\to\infty} \frac{10 - \frac{45}{x} + \frac{35}{x^2}}{12 - \frac{60}{x} + \frac{48}{x^2}} = \frac{10}{12} = \frac{5}{6}$$

Note: If you check the limit as $x \to -\infty$, the result is again $\frac{5}{6}$. This function has only one horizontal asymptote

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5. When $x < 0$, it is true that $\dfrac{x - x^2 + x^4}{x - x^2} \leq \dfrac{1 - x + x^3\cos(\frac{2}{x})}{1 - x} \leq \dfrac{1 - x - x^3}{1 - x}$
What is $\displaystyle\lim_{x\to 0^-} \dfrac{1 - x + x^3\cos(\frac{2}{x})}{1 - x}$?

A) -4 B) -2 C) -1 D) 0   **E) 1**
F) 2   G) 3   H) 4   I) 5   J) not enough information to determine an answer

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As $x \to 0^-$,

$$\frac{x - x^2 + x^4}{x - x^2} = \frac{1 - x + x^3}{1 - x} \to 1 \text{ and } \frac{x - x^2 + x^4}{x - x^2} = \frac{1 - x - x^3}{1 - x} \to 1$$

so, by the Squeeze Theorem, $\dfrac{1 - x + x^3\cos(\frac{2}{x})}{1 - x} \to 1$

<u>Alternate solution:</u> $-1 \leq \cos(\frac{2}{x}) \leq 1$ for all x. Since $x \to 0^-$, $x^3 < 0$, so

$$-x^3 \geq x^3\cos(\tfrac{2}{x}) \geq x^3 \text{ so } x^3 \leq -x^3\cos(\tfrac{2}{x}) \leq -x^3.$$

As $x \to 0^-$ both x^3 and $-x^3 \to 0$, so by the Squeeze Theorem, $-x^3\cos(\frac{2}{x}) \to 0$, so

$x^3\cos(\frac{2}{x}) \to 0$. Therefore $\displaystyle\lim_{x\to 0^-} \dfrac{1 - x + x^3\cos(\frac{2}{x})}{1 - x} = \dfrac{1 - 0 + 0}{1 - 0} = 1$

6. What is $\lim\limits_{t \to \infty} \dfrac{3 - 2t + 7t^2}{4 + 3t + 5t^2}$?

A) $\frac{2}{3}$ B) $\frac{7}{3}$ C) $\frac{2}{5}$ **D) $\frac{7}{5}$**   E) $\frac{3}{4}$

F) $-\frac{3}{5}$ G) $-\frac{1}{2}$ H) $-\frac{7}{4}$ I) -1 J) DNE

$$\lim_{t \to \infty} \frac{3 - 2t + 7t^2}{4 + 3t + 5t^2} = \lim_{t \to \infty} \frac{\frac{3}{t^2} - \frac{2}{t} + 7}{\frac{4}{t^2} + \frac{3}{t} + 5} = \frac{0 - 0 + 7}{0 + 0 + 5} = \frac{7}{5}$$

7. Let $f(x) = \dfrac{x^2 + x + 6}{x}$. The Intermediate Value Theorem guarantees that which of the following equations has a solution in the interval $[1, 3]$?

A) $f(x) = -6.5$ B) $f(x) = -4$ C) $f(x) = -2.5$ D) $f(x) = 0$

E) $f(x) = 2.5$ F) $f(x) = 4$ G) $f(x) = 5.5$ **H) $f(x) = 6.5$**

I) $f(x) = 8.5$ J) $f(x) = 9.5$

$f(1) = \frac{1 + 1 + 6}{1} = 8$ and $f(3) = \frac{9 + 3 + 6}{3} = 6$. The Intermediate Value Theorem

guarantees that $f(x) = N$ will have a solution whenever N is a number between 8 and 6.

H) is the only choice that works.

8. Suppose f and g are continuous functions and that $g(2) = 4$.

If $\lim\limits_{x \to 2} \left[f(x)g(x) + \dfrac{f(x)}{g(x)} \right] = 17$, then what is $f(2)$?

A) 4   B) 3   C) 2   D) 1   E) 0

F) -1 G) -2 H) -3 I) -4 J) Not enough information
                                                           to determine an answer

By continuity, $\lim\limits_{x \to 2} f(x) = f(2)$ and $\lim\limits_{x \to 2} g(x) = g(2) = 4$.

So $\lim\limits_{x \to 2} \left[f(x)g(x) + \dfrac{f(x)}{g(x)} \right] = \lim\limits_{x \to 2} f(x) \lim\limits_{x \to 2} g(x) + \dfrac{\lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)} = f(2)g(2) + \dfrac{f(2)}{g(2)}$

$= 4f(2) + \dfrac{f(2)}{4} = 17.$ Therefore $17f(2) = 68$, so $f(2) = 4$.

9. Find $\lim\limits_{t \to 2^+} \dfrac{4|2-t|}{(2-t)|t+6|}$

A) 0   B) 1   C) $\frac{1}{2}$ D) $\frac{1}{6}$ E) $\frac{1}{8}$

F) -1 **G) $-\frac{1}{2}$**   H) $-\frac{1}{6}$ I) $-\frac{1}{8}$ J) DNE

As $t \to 2^+$: $t > 2$, so $2 - t < 0$ and $|2 - t| = t - 2$

$t + 6 > 0$, so $|t + 6| = t + 6$

Therefore, $\lim\limits_{t \to 2^+} \dfrac{4|2-t|}{(2-t)|t+6|} = \lim\limits_{t \to 2^+} \dfrac{4(t-2)}{(2-t)(t+6)} = \lim\limits_{t \to 2^+} \dfrac{-4}{(t+6)} = -\dfrac{4}{8} = -\dfrac{1}{2}$

10. The function $y = f(x) = $ has <u>how many</u> vertical asymptotes?

A) 0   **B) 1**   C) 2   D) 3   E) 4   F) 5

(There are only 5 choices for this answer: nothing is wrong with your exam copy!)

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{(\sin x)(x-1)^3 (x-2)^2 (x-3)}{x(x-1)^2 (x-2)^3 \sqrt[3]{(x-3)}}$$

$$= \lim_{x \to a} \frac{\sin x}{x} \frac{(x-1)(x-3)^{2/3}}{(x-2)} = 1 \cdot \lim_{x \to a} \frac{(x-1)(x-3)^{2/3}}{(x-2)}.$$

This limit exists except for $a = 2$. When $a = 2$,

$$\lim_{x \to 2^-} \frac{(x-1)(x-3)^{2/3}}{(x-2)} = -\infty \text{ and } \lim_{x \to 2^+} \frac{(x-1)(x-3)^{2/3}}{(x-2)} = \infty$$

There is <u>one</u> vertical asymptote: $x = 2$

11. Find $\displaystyle\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$

A) 0   B) $\frac{1}{12}$ **C) $\frac{1}{6}$**   D) $\frac{1}{3}$ E) $\frac{1}{8}$

F) $\frac{1}{9}$ G) 1   H) 3   I) 9   J) DNE

$$\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h \to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{9+h}+3)} = \lim_{h \to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

12. What is $\lim\limits_{x\to 0} \frac{\sin 3x}{4x}$?

A) 0   B) 1   C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) $\frac{3}{4}$

F) $\frac{1}{3}$ G) $\frac{4}{3}$ H) 3   I) 4   J) DNE

Substitute $u = 3x$ (so that $x = \frac{u}{3}$). Then "$x \to 0$" is the same as syaing "$u \to 0$."

So $\lim\limits_{x\to 0} \frac{\sin 3x}{4x} = \lim\limits_{u\to 0} \frac{\sin u}{4(\frac{u}{3})} = \lim\limits_{x\to 0} \frac{3}{4}\frac{\sin u}{u} = \frac{3}{4}(1) = \frac{3}{4}$.

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13. The function $f(x) = \dfrac{\frac{1}{x} - \frac{1}{3}}{(x-3)}$ is not defined when $x = 3$. Therefore $f(x)$ is not continuous at 3. But the discontinuity at 3 is a removable discontinuity: that is, we can define $f(3) = c$ to create a function that is continuous at 3. What is c?

A) $\frac{1}{8}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{2}$ E) 0

F) $-\frac{1}{3}$ G) $-\frac{1}{5}$ H) $-\frac{1}{7}$ I) $-\frac{1}{9}$ J) $-\frac{1}{11}$

We need to define $f(3) = c = \lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{(x-3)} = \lim\limits_{x\to 3} \dfrac{\frac{3-x}{3x}}{(x-3)}$

$= \lim\limits_{x\to 3} \dfrac{3-x}{3x(x-3)} = \lim\limits_{x\to 3} \dfrac{-(x-3)}{3x(x-3)} = \lim\limits_{x\to 3} -\dfrac{1}{3x} = -\dfrac{1}{9}$

14. Find $\displaystyle\lim_{x \to -\infty} \frac{5x}{\sqrt{4x^2 + 2x + 2}}$

A) $\frac{5}{4}$ B) $\frac{5}{2}$ C) $\frac{5}{3}$ D) $\frac{5}{\sqrt{2}}$ E) 0

F) $-\frac{5}{4}$ **G) $-\frac{5}{2}$**   H) $-\frac{5}{3}$ I) $-\frac{5}{\sqrt{2}}$ J) DNE

$$\lim_{x \to -\infty} \frac{5x}{\sqrt{4x^2 + 2x + 2}} = \lim_{x \to -\infty} \frac{5}{\frac{\sqrt{4x^2 + 2x + 2}}{x}} = \lim_{x \to -\infty} \frac{5}{-\sqrt{\frac{4x^2 + 2x + 2}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{5}{-\sqrt{4 + \frac{2}{x} + \frac{2}{x^2}}} = -\frac{5}{\sqrt{4}} = -\frac{5}{2}$$

Questions 15) - 19) are "true/false" questions (worth 1 point each).

15. $\displaystyle\lim_{x \to \infty}(x^2 - x) = 0$

False: $x^2 - x = x(x - 1)$.   As $x \to \infty$, both x and $x - 1 \to \infty$, so the product $x(x - 1) \to \infty$.

A) True   **B) False**

16. If $\displaystyle\lim_{x \to 0} f(x) = 0$ and $\displaystyle\lim_{x \to 0} g(x) = 0$, then $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)}$ cannot exist.

False: For example, if $f(x) = x^2$ and $g(x) = x$, the both $f(x)$ and $g(x) \to 0$ as $x \to 0$, and $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$.

A) True   **B) False**

17. The graph of a function $y = f(x)$ can have <u>at most</u> 2 horizontal asymptotes.

> True: the horizontal asymptotes are found by calculating $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.
>
> The only possibilities are: both limits exist (2 horizontal asymptotes), only one of the limits exists (only one horizontal asymptote) or neither limit exists (no horizontal asymptotes).

A) True   B) False

18. If $x = 1$ is a vertical asymptote of $y = f(x)$, then f cannot be defined at $x = 1$.

> False. For example, look at the function $f(x) = \begin{cases} \frac{1}{x} & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$

A) True   **B) False**

19. Suppose $f(-1) = 4$, $f(1) = 3$ and that the function $f(x)$ is continuous on the interval $[-1, 1]$. Then there must be a number r such that $|r| < 1$ and $f(r) = \pi$.

> True: Since $f(x)$ is continuous on the interval $[-1, 1]$, the Intermediate Value Theorem applies: π is a number between $f(-1)$ and $f(1)$, so there must be an r in $[-1, 1]$ for which $f(r) = \pi$. Since we know that $r = -1$ and $r = 1$ don't work, the r that does work must satisfy $|r| < 1$.

A) True   B) False

PART II (25 points)

Name _____ WUSTL ID _____

Circle the <u>time</u> of your discussion section on the line that has <u>your TA's name</u>:

Mr. Gong Cheng,   **8 a.m.**   **9 a.m.**   **10 a.m.**   **11 a.m.**

Mr. Cody Stockdale, **8 a.m.**   **9 a.m.**   **10 a.m.**   **12 a.m.**

Ms. Wei Wang,   **10 a.m.**   **11 a.m.**   **12 a.m.**

For each problem, clearly show your solution in the space provided. "Show your solution" does not simply mean "show your scratch work" — you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. <u>A correct answer without supporting work may not receive full credit</u>.

1. Suppose $f(x) = \frac{1}{x+1}$ $\qquad$ $g(x) = \frac{x}{x+1}$ $\qquad$ $h(x) = \frac{x}{1-x}$

For each part, simplify your answer as much as possible. Put your final answer in the box provided.

a) What is $g(h(x))$?

$$g(h(x)) = \frac{h(x)}{h(x)+1} = \frac{\frac{x}{1-x}}{\frac{x}{1-x}+1} = \frac{\frac{x}{1-x}}{\frac{x}{1-x}+\frac{1-x}{1-x}} = \frac{\frac{x}{1-x}}{\frac{1}{1-x}} = x$$

Final simplified answer:

$g(h(x)) = x$

b) What is $\frac{f(x+h)-f(x)}{h}$?

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} = \frac{\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h} = \frac{\frac{-h}{(x+h+1)(x+1)}}{h}$$

$$= \frac{-h}{h(x+h+1)(x+1)} = \frac{-1}{(x+h+1)(x+1)}$$

Final simplified answer:

$\frac{f(x+h)-f(x)}{h} = \frac{-1}{(x+h+1)(x+1)}$

2. On the grid below, sketch the graph of a function $y = f(x)$ that <u>satisfies all</u> of the conditions listed below. *Use a solid dot "●" or an open dot "o" to emphasize, where necessary, that a point is or is not on the graph.*

$f(x)$ is defined for <u>EVERY</u> x

$\lim\limits_{x \to 0^+} f(x)$ exists

$\lim\limits_{x \to 2} f(x)$ exists but f is <u>not</u> continuous at $x = 2$

$\lim\limits_{x \to -\infty} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = -\infty$

$\lim\limits_{x \to 2^+} f(x) = 4$

$\lim\limits_{x \to \infty} f(x) = 8$

$f(-2) = 0$

The Intermediate Value Theorem does <u>not</u> apply to the function $f(x)$ on the interval $[-6, -2]$

**(When you draw, <u>use the scale given on each of the axes.</u>)**
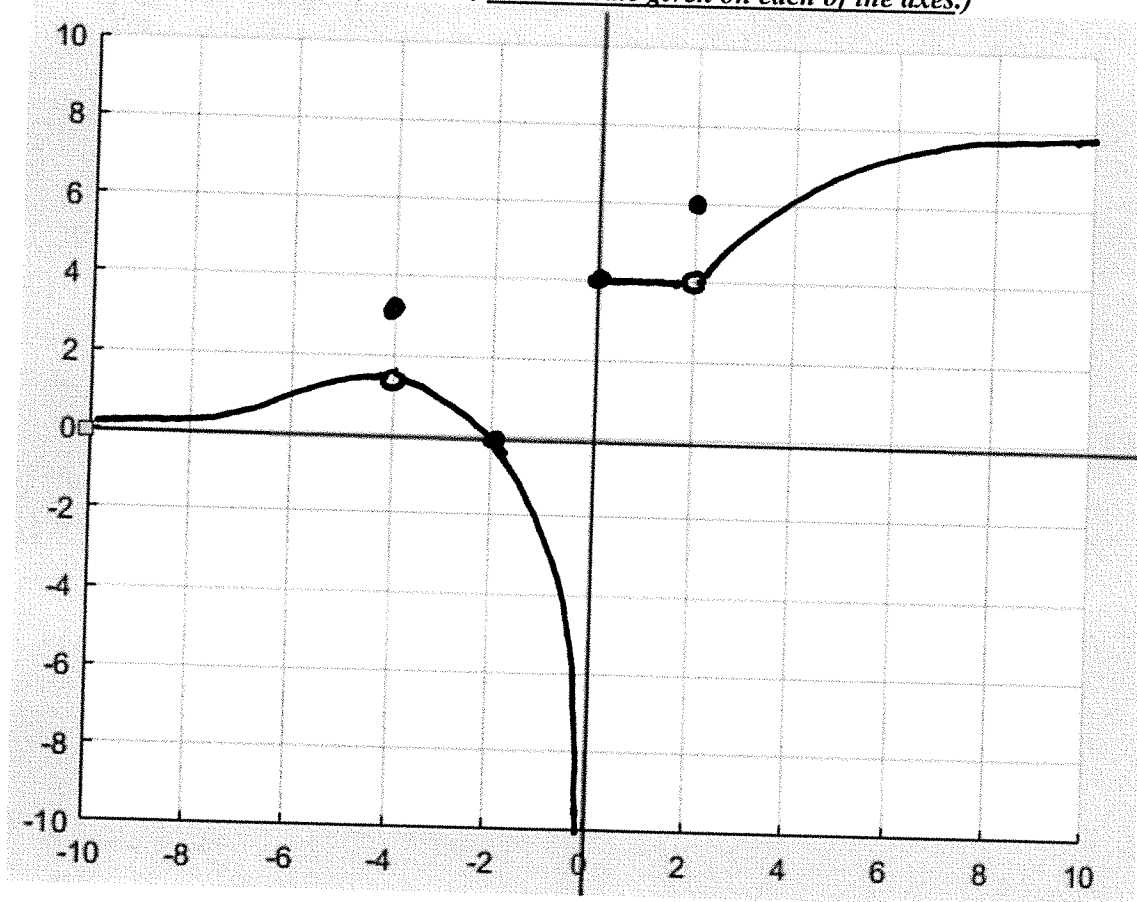
3. Suppose $y = f(x) = \begin{cases} A^2 x^2 & \text{if } x \leq 2 \\ (1-A)x & \text{if } x > 2 \end{cases}$

a) Draw the graph of $f(x)$ when $A = -1$.

When $A = -1$, $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$

(The graph consists of part of the parabola $y = x^2$ and part of the straight line $y = 2x$. The "switch" from parabola to straight line happens at $x = 1$, and the two pieces meet at $(2, 4)$ to create one continuous function.)

(Notice the scales given on the axis when you draw)
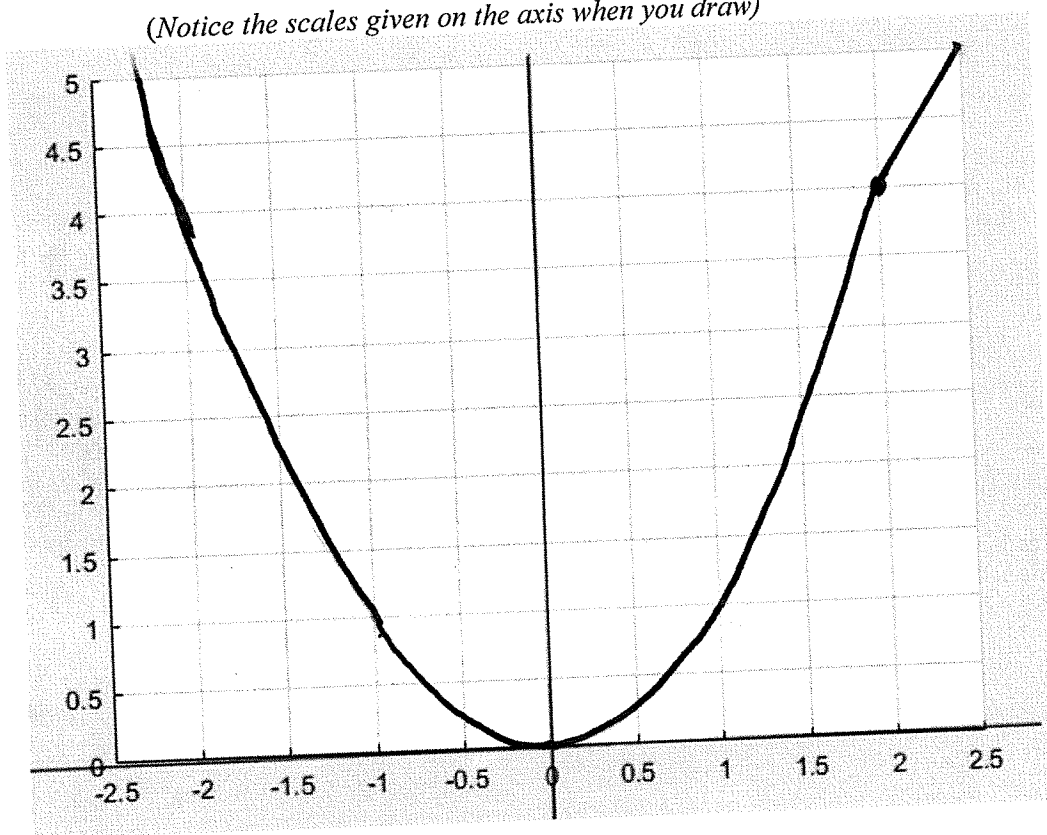
b) Find all the value(s) of A that will make $f(x)$ continuous at 2. Put your final answer in the box provided.

For f to be continuous at 2, we need to have $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

$$\lim\limits_{x \to 2^-} f(x) = 4A^2 = 2(1-A) = \lim\limits_{x \to 2^+} f(x)$$

so

$$4A^2 = 2 - 2A$$

$$2A^2 + A - 1 = 0$$

$$(2A - 1)(A + 1) = 0$$

so the left and right limit will be equal when $A = -1$ or $A = \frac{1}{2}$:

For $A = -1$: $\quad f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$

$\lim\limits_{x \to 2^-} f(x) = 4 = \lim\limits_{x \to 2^+} f(x)$, and we can see from the formula that it is

also true that $f(2) = 4$

so $A = -1$ works *(as should be clear in your graph, above)*

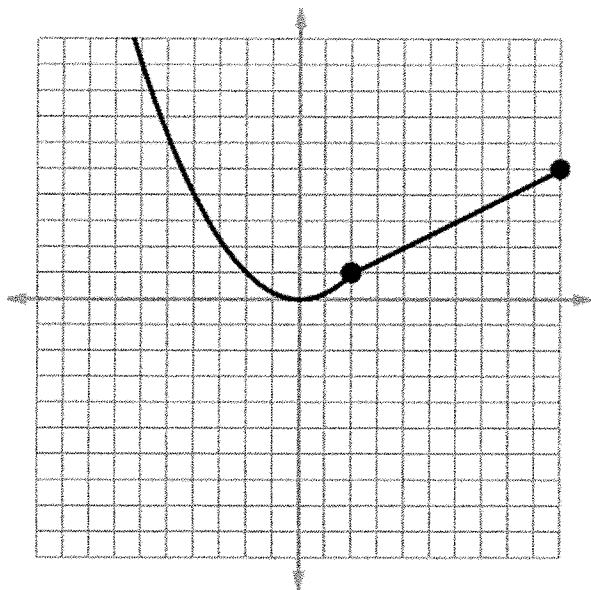For $A = \frac{1}{2}$: $\quad f(x) = = \begin{cases} \frac{1}{4}x^2 & \text{if } x \leq 2 \\ \frac{1}{2}x & \text{if } x > 2 \end{cases}$

$\lim\limits_{x \to 2^-} f(x) = 1 = \lim\limits_{x \to 2^+} f(x)$, and we can see from the formula that it is

also true that $f(2) = 1$.

So $A = \frac{1}{2}$ also works

Final answer: the value(s) of A are: -1 and $\frac{1}{2}$

(Not required: here is a picture of the graph when $A = \frac{1}{2}$:

Part II Scores

Q1:   / 8

Q2:   / 9

Q3:   / 8

TOTAL PART II: /25