Math 131 **Exam 2, FL2016**

Solutions

Name ________________________________ **WU ID Number** __________

No calculators or notes are allowed on any part of this exam. As usual, "DNE" stands for "does not exist." You may have a single 3x5 card with notes.

This exam should have 9 white pages (Part I), and 5 colored pages (Part II). The pages in each part are numbered in the upper right corner.

Part I consists of 14 multiple choice questions (worth 5 points each) and 5 true/false questions (worth 1 point each), for a total of 75 points. Mark the correct answer on the answer card. You may use the test booklet for any work you want to write down, but for Part I, only the answer marked on the answer card will be graded.

Part II (separate colored pages) consists of 3 hand-graded questions, worth a total of 25 points. Also put your name on Part II, write your solutions on the colored sheets. Your written work will be scored by the instructor and TAs.

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1. The point $P = (2, 35)$ is on the graph of $f(x) = 3x^2 + \frac{46}{x}$. What is the equation of the line perpendicular to the tangent line to the graph at P?

A) $\boldsymbol{y = -2x + 39}$ B) $y = 2x + 31$ C) $y = \frac{1}{2}x + 34$

D) $y = -\frac{1}{2}x + 36$ E) $y = x + 33$ F) $y = -2x + 36$

G) $y = 4x + 27$ H) $y = -\frac{1}{4}x + 37$ I) $y = 5x + 25$

J) $y = 3x + 29$

$f'(x) = 6x - \frac{46}{x^2}$, so $f'(2) = 12 - \frac{46}{4} = \frac{48}{4} - \frac{46}{4} = \frac{2}{4} = \frac{1}{2}$. This is the slope of the tangent line at $(2, 35)$, so the slope of the perpendicular ("normal") line is -2. So the normal line has equation $y - 35 = -2(x-2)$, or $y = -2x + 39$.

2. Suppose $h(x) = \sin(\pi f(x))$ and that

$$f(0) = 1 \qquad f'(0) = 3$$

What is $h'(0)$?

A) -4π **B)** $\mathbf{-3\pi}$ C) -2π D) π E) 0

F) π G) 2π H) 3π I) 4π J) 5π

$h'(x) = \cos(\pi f(x)) \cdot (\pi f(x))' = \pi \cos(\pi f(x)) \cdot f'(x)$,

so $h'(0) = \pi \cos(\pi f(0)) f'(0) = \pi \cos(\pi) \cdot (3) = -3\pi$

3. The point $P = (1, 0)$ is on the graph of $xe^y + y = 1$. What is the slope of the tangent line at P?

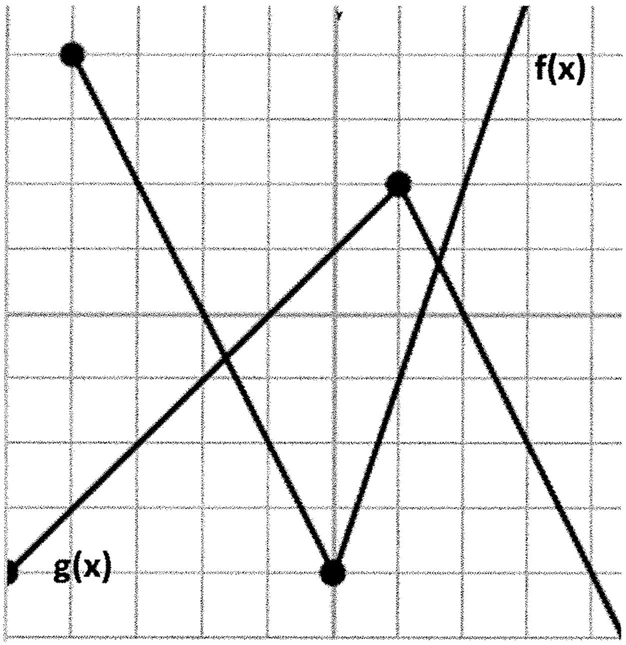
A) 0 B) 1 C) 2 D) e E) e^2

F) $-e$ **G)** $-\frac{1}{2}$ H) $-e^2$ I) $2e$ J) $\frac{1}{4}$

Differentiating implicitly gives: $xe^y y' + (1)e^y + y' = 0$. Substituting $(1, 0)$ gives

$$(1)(1)y' + (1)(1) + y' = 0$$
$$2y' + 1 = 0$$
$$y' = -\frac{1}{2}$$

4. The graph shows two functions $f(x)$ and $g(x)$:



Let $h(x) = f(x)g(x)$. What is $h'(2)$?

A) 5 B) 4 C) 3 D) 2 E) 1

F) 0 G) -1 H) -2 I) -3 **J) -4**

$$h'(x) = f(x)g'(x) + f'(x)g(x), \text{ so } h'(2) = f(2)g'(2) + f'(2)g(2)$$
$$= (2)(-2) + f'(2)(0) = -4$$

5. If $y = g(x) = \log_{10}(x^3 + x^2 + 2x + 1)$, what is $g'(1)$?

A) $\frac{7}{50}$ B) $\frac{1}{5\ln(10)}$ **C) $\frac{7}{5\ln 10}$** D) $\frac{7}{5}$ E) $\frac{1}{5}$

F) $\frac{1}{2\ln(10)}$ G) $\frac{7}{5\log_{10}(10)}$ H) $\frac{1}{5\log_{10}(10)}$ I) 0 J) 1

$$g'(x) = \frac{1}{\ln(10)\,(x^3+x^2+2x+1)} \cdot (3x^2+2x+2), \text{ so } g'(1) = \frac{7}{5\ln(10)}$$

6. A particle is moving along the x-axis. Its position at time t is $s = f(t) = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t + 1$. For what times t is the particle moving to the left?

A) $t > 0$ B) $t > 1$ C) $0 < t < 2$ **D) $\mathbf{1 < t < 4}$**

E) $t > 4$ F) $-1 < t < 0$ G) $-2 < t$ H) $-4 < t < -1$

I) $-2 < t < 2$ J) $-4 < t < 4$

The particle is moving left when its velocity $v < 0$.

$v = \frac{ds}{dt} = t^2 - 5t + 4 = (t-1)(t-4)$

for $t < 1$, both factors are negative so $v > 0$
for $t > 4$, both factors are positive so $v > 0$
when $1 < t < 4$: $(t-1) > 0$ and $(t-4) < 0$ so $v < 0$

The particle is moving left for $1 < t < 4$.

7. What is $\lim_{h \to 0} \frac{\arctan(2+h) - \arctan(2)}{h}$?

A) 0 B) 1 C) 2 D) 4 E) π

F) $\frac{\pi}{4}$ G) $\frac{\pi}{2}$ H) $-\frac{1}{2}$ I) $\frac{1}{5}$ J) DNE

You need to recognize that this limit is a derivative: if $f(x) = \arctan x$, then $f'(2) = \lim_{h \to 0} \frac{\arctan(2+h) - \arctan(2)}{h}$. But we know that $f'(x) = \frac{1}{1+x^2}$.
and therefore $f'(2) = \frac{1}{1+2^2} = \frac{1}{5}$.

8. We start with 10 g of a radioactive substance unobtainium. It decays into other isotopes at a rate proportional to the amount the amount of unobtainium present. After 1 minute, only 5g of unobtainium remains. How much unobtainium remains after 3.4 minutes ?

A) $2^{-3.4}$ g B) $2^{-6.8}$ g C) $2^{-1.7}$ g D) $5 \cdot 2^{-3.4}$ g

E) $5 \cdot 2^{-6.8}$ g F) $5 \cdot 2^{-1.7}$ g **G) $\mathbf{10 \cdot 2^{-3.4}}$ g** H) $10 \cdot 2^{-6.8}$ g

I) $10 \cdot 2^{-1.7}$ g J) $\frac{e}{2}$ g

The mass at time t satisfies $\frac{dm}{dt} = kt$, so $m = m_0 e^{kt} = 10e^{kt}$.

After 1 minute, only 5g of the substance remains: $5 = 10e^{k \cdot 1}$. Therefore $\frac{1}{2} = e^k$, so $k = \ln(\frac{1}{2}) = \ln(1) - \ln(2) = -\ln(2)$.

$m = 10e^{-\ln(2)t}$, so after 3.4 minutes, $m = 10e^{-\ln(2)(3.4)} = 10 \cdot 2^{-3.4}$.

9. What is $\lim_{x \to 0} \frac{\sin 2x \sin 6x}{8x^2}$?

A) $\frac{1}{4}$ B) $\frac{3}{4}$ **C) $\frac{3}{2}$** D) 1 E) $\frac{5}{2}$

F) 2 G) 6 H) 8 I) 12 J) DNE

$$\lim_{x \to 0} \frac{\sin 2x \sin 6x}{8x^2} = \lim_{x \to 0} \frac{\sin 2x}{8x} \cdot \frac{\sin 6x}{x} = \lim_{x \to 0} \frac{\sin 2x}{8x} \cdot \lim_{x \to 0} \frac{\sin 6x}{x}.$$

Let $h = 2x$, so $8x = 4h$. As $x \to 0$, then $h \to 0$ also. So

$\lim_{x \to 0} \frac{\sin 2x}{8x} = \lim_{h \to 0} \frac{\sin h}{4h} = \frac{1}{4} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{4}$. A similar substitution shows that

$\lim_{x \to 0} \frac{\sin 6x}{x} = 6$. So $\lim_{x \to 0} \frac{\sin 2x \sin 6x}{8x^2} = \frac{1}{4}(6) = \frac{3}{2}$

10. For times $t > 0$ (sec), a particle moving along a straight line has position $s = \frac{3t^3 - 2t^2 + t}{t}$ (ft). What is its velocity at time $t = 2$?

A) 0 ft/sec B) $\frac{1}{4}$ ft/sec C) 2 ft/sec D) $\frac{10}{4}$ ft/sec E) 3 ft/sec

F) $\frac{35}{4}$ ft/sec **G) 10 ft/sec** H) 11 ft/sec I) $\frac{23}{2}$ ft/sec J) 13 ft/sec

Simplify first to save work: $s = \frac{3t^3 - 2t^2 + t}{t} = 3t^2 - 2t + 1$, so
$v = \frac{ds}{dt} = 6t - 2.$

At time $t = 2$, $v = 10$ft/sec.

11. If $f(t) = \ln\left(\frac{t^2}{(t+1)\,e^t}\right)$, what is $f'(1)$?

A) 0 B) 1 C) 2 D) 3 E) 4

F) $\frac{1}{2}$ G) $\frac{1}{3}$ H) $\frac{1}{4}$ I) $\frac{1}{5}$ J) -1

$$f(t) = \ln\left(\frac{t^2}{(t+1)\,e^t}\right) = \ln(t^2) - \ln\Big((t+1)(e^t)\Big) = 2\ln t - \ln(t+1) - \ln(e^t)$$

$$= 2\ln t - \ln(t+1) - t.$$

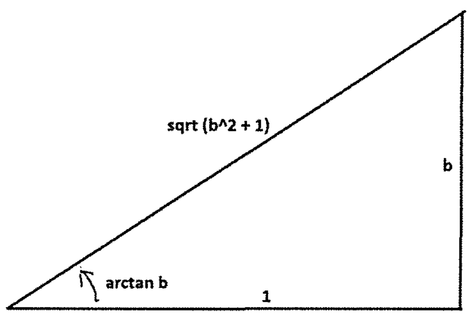
So $f'(t) = \frac{2}{t} - \frac{1}{t+1} - 1$ and $f'(1) = \frac{2}{1} - \frac{1}{1+1} - 1 = \frac{1}{2}$.

12. Simplify $\cos^2(\arctan b)$

A) b^2 B) $1+b^2$ C) $2b^2$ **D)** $\mathbf{\frac{1}{1+b^2}}$ E) $\frac{b}{1+b^2}$

F) $\frac{b^2}{1+b^2}$ G) $\frac{b}{b^2-1}$ H) $\frac{4b}{4+b^2}$ I) $\frac{b}{\sqrt{1+b^2}}$ J) $\frac{2b}{\sqrt{1+2b^2}}$

Draw a right triangle illustrating "arctan b" = "angle with tangent b":



Then $\cos(\arctan b) = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{\sqrt{b^2+1}}$

so $\cos^2(\arctan b) = \frac{1}{1+b^2}$. *The picture illustrates the case where* $\arctan b$ *is positive. But a similar picture illustrates the case with a negative* $\arctan b$.

<u>or</u>

You can avoid pictures and use a trig identity:

$\sec^2(\arctan b) = 1 + \tan^2(\arctan b) = 1 + b^2$, so $\cos^2(\arctan b)$

$$= \frac{1}{\sec^2(\arctan b)} = \frac{1}{1+b^2}$$

13. If $f(x) = \sin^2 x$, what is $f'(\frac{\pi}{6})$?

A) $\mathbf{\frac{\sqrt{3}}{2}}$ B) $\frac{\sqrt{2}}{2}$ C) $\frac{1}{2}$ D) 1 E) 0

F) $-\frac{\sqrt{3}}{2}$ G) $-\frac{\sqrt{2}}{2}$ H) $-\frac{1}{2}$ I) $-\frac{1}{3}$ J) $-\frac{1}{4}$

$f'(x) = 2\sin x\cos x$, so $f'(x) = 2\sin(\frac{\pi}{6})\cos(\frac{\pi}{6}) = 2(\frac{1}{2})(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2}$.

14. The function $y = f(x) = \ln\left(\frac{4a+x^2}{e^x}\right)$ has a horizontal tangent line at $x = 1$. What is the value of a ?

A) 4 B) 2 C) $\frac{1}{2}$ **D)** $\mathbf{\frac{1}{4}}$ E) 0

F) $\frac{5}{e}$ G) $\frac{1}{e}$ H) $\frac{4}{e^2}$ I) $-\frac{1}{2}$ J) -2

> $f(x) = \ln(4a + x^2) - \ln(e^x) = \ln(4a + x^2) - x$
>
> $f'(x) = \frac{2x}{4a+x^2} - 1$. Since we know $f'(1) = 0$, we have $\frac{2(1)}{4a+1} - 1 = 0$.
>
> Therefore $\frac{2-4a-1}{4a+1} = 0$, so $-4a + 1 = 0$ and $a = \frac{1}{4}$.

Questions 15 - 19 are "true/false" questions (worth 1 point each)

15. $\frac{d}{dx}\ln(\pi) = \frac{1}{\pi}$

A) True **B) False**

> False: $\ln(\pi)$ is a constant, so $\frac{d}{dx}\ln(\pi) = 0$.

16. If $f(x)$ is not differentiable at a, then $f(x)$ cannot be continuous at a.

A) True **B) False**

> False: for example, $f(x) = |x|$ is not differentiable at $a = 0$, but $f(x)$ IS continuous at $a = 0$.

17. If $f'(x) = g'(x)$ for every x, then it must be that $f(x) = g(x)$ for every x.

A) True **B) False**

> False: for example, let $f(x) = x^2$ and $g(x) = x^2 + 1$.

18. A cubic polynomial function $y = ax^3 + bx^2 + cx + d$ must have at least one horizontal tangent line to its graph.

A) True **B) False**

False: The derivative is a quadratic function: $y' = 3ax^2 + 2bx + c$. A horizontal tangent line would occur where $3ax^2 + 2bx + c = 0$. But (depending on the values of a, b, c) this quadratic equation <u>might</u> not have any (real) solutions.

19. A point moves along the x-axis with position $s = t^3 - 3t^2$ at time t. For $0 < t < 1$, its speed is increasing.

A) True B) False

True: $v = 3t^2 - 6t = 3t(t-2)$: when $0 < t < 1$, we see that $v < 0$
$a = 6t - 6 = 6(t-1)$: when $0 < t < 1$, we see that $a < 0$
Since both v and a are <u>both</u> negative when $0 < t < 1$, the speed is increasing (in the negative direction).

PART II (25 points)

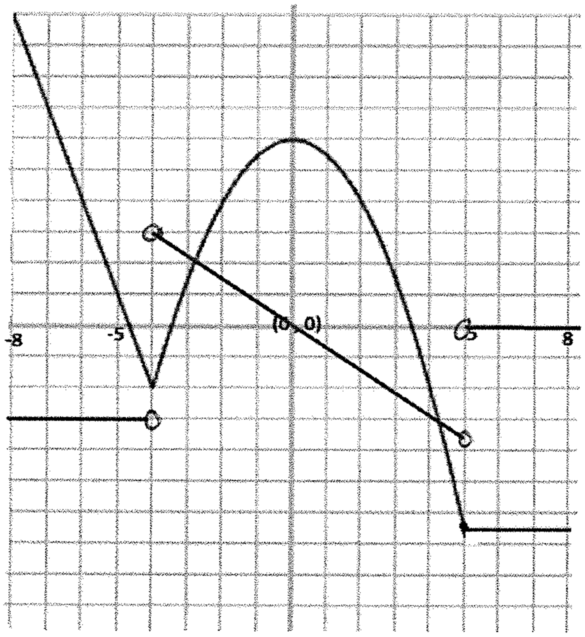
Name ________________________________ **WUSTL ID** ____________

Circle the <u>time</u> of your discussion section on the line that has <u>your TA's name</u>:

Mr. Gong Cheng,	**8 a.m.**	**9 a.m.**	**10 a.m.**	**11 a.m.**	
Mr. Cody Stockdale,	**8 a.m.**	**9 a.m.**	**10 a.m.**		**12 p.m.**
Ms. Wei Wang,			**10 a.m.**	**11 a.m.**	**12 p.m.**

For each problem, clearly show your solution in the space provided. "Show your solution" does not simply mean "show your scratch work" – you should cross out any scratch work that turned out to be wrong or irrelevant and, where appropriate, present a readable, orderly sequence of steps showing how you got the answer. <u>A correct answer without supporting work may not receive full credit</u>.

1. Here is the graph of a function $y = f(x)$. The grid lines on both axes in the picture are one unit apart.



a) For $-8 < x < 8$:

List all the x's for which $f'(x)$ does not exist: $\quad$ $\boldsymbol{x = -4,\ 5}$

For what x's is $f'(x) > 0$? $\quad$ **$f'(x) > 0$ for those x's where the slope of the tangent line is positive: that happens only for $-4 < x < 0$.**

For what x's is $f'(x) = 0$? $\quad$ **$f'(x) = 0$ where $x = 0$ and where $x > 5$**

b) On the same grid, draw a reasonable graph for $y = f'(x)$. Use an solid dot • where needed to emphasize that a point is on the graph, and an open dot o to indicate that a.point is not included on the graph.

2. Find each derivative. You do <u>not</u> need to simplify after you have finished differentiating, BUT

↓

DRAW A BOX AROUND YOUR FINAL ANSWER IN EACH PART

a) Is $s = f(t) = 2(t^2 - 3t)^{23}$, then

$$\frac{ds}{dt} = (2)(23)(t^2 - 3t)^{22}(2t - 3) = 46(t^2 - 3t)^{22}(2t - 3)$$

b) If $y = f(x) = \arcsin(3x^2) + \arctan(x^2)$, then

$$f'(x) = \frac{1}{\sqrt{1-(3x^2)^2}}(6x) + \frac{1}{1+(x^2)^2}(2x) = \frac{1}{\sqrt{1-9x^4}}(6x) + \frac{1}{1+x^4}(2x)$$

c) If $y = \tan(\sin x) + 3^{x^2}$, then

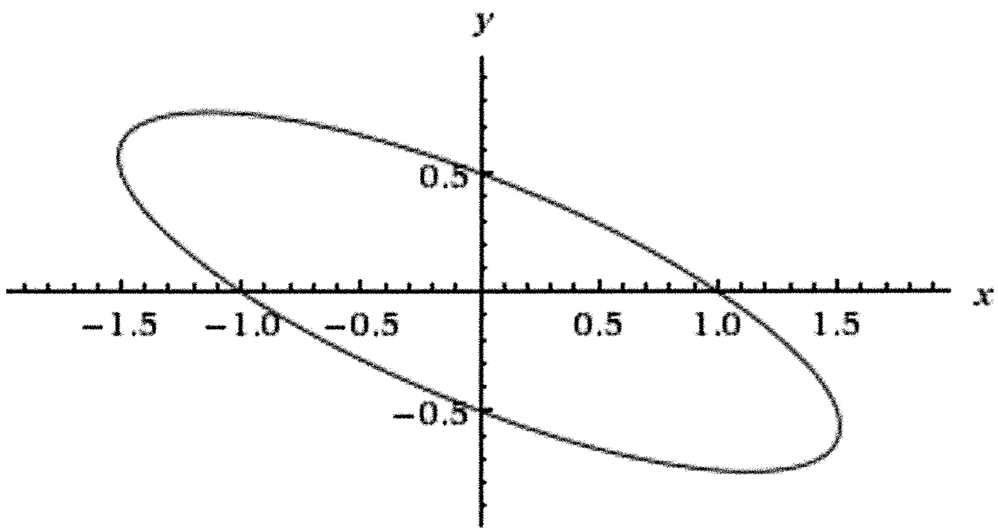
$$f'(x) = \sec^2(\sin x)(\cos x) + \ln(3)\cdot 3^{x^2}(2x)$$

d) If $y = x^{\cos x}$, then $\ln(y) = \ln(x^{\cos x}) = \cos x \ln x$.

Therefore $\frac{y'}{y} = (\cos x)\cdot\frac{1}{x} - (\sin x)\ln x$, so

$$y' = y\Big((\cos x)\cdot\frac{1}{x} - (\sin x)\ln x\Big) = x^{\cos x}\Big((\cos x)\cdot\frac{1}{x} - (\sin x)\ln x\Big)$$

3. The picture shows the graph of $x^2 + 3xy + 4y^2 = 1$. It is an ellipse that contains the point $(1, 0)$.



a) Let m be the slope of the tangent line at $(1, 0)$. Write the equation of the tangent line (in terms of m). Put your final answer in the box below.

Tangent line at $(1, 0)$: $(y - 0) = m(x - 1)$ or $y = mx - m$

b) m is the value of the derivative $y' = \frac{dy}{dx}$ at the point $(1, 0)$. Find the value of m. Put your final value for m in the box below.

$x^2 + 3xy + 4y^2 = 1$. Differentiate both sides with respect to x to get:

$2x + 3xy' + 3y + 8yy' = 0.$ Setting $x = 1$ and $y = 0$ we get

$2 \quad + 3y' \quad + 0 \quad + 0 \quad = 0$, so $y' = -\frac{2}{3}$

Final answer: $m = -\frac{2}{3}$

DO NOT WRITE ON THIS PAGE. IT IS FOR SCORING PURPOSES ONLY:

Part II Scores

Q1:	/ 9

Q2:	/ 8

Q3:	/ 8

TOTAL PART II: /25