Suppose f is a continuous function defined on a closed interval $[a, b]$. To define the <u>definite integral</u> $\int_a^b f(x)\,dx$:

Divide $[a, b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. The subintervals are $a = [x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{i-1}, x_i],\ \ldots,\ [x_{n-1} x_n = b]$

Pick a "sample point" x_i^* in the i^{th} interval for each $i = 1, 2, \ldots, n$ (*Choose any way you like: the sample points could be left endpoints, right endpoints, midpoints, or even some randomly chosen point in each interval not following any particular rule.*) Then form the Riemann sum : $\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Take the limit as $n \to \infty$ (i.e., as $\Delta x \to 0$): $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$

(*Because f is continuous, it can be proven that the limit must exist, and that it is always the same no matter how you chose to pick the sample points* x_i^*)

Since $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$, a Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is an approximation for $\int_a^b f(x)\,dx$ and, generally, the bigger the n, the better the approximation.

Suppose $x =$ time (sec) and $f(x) =$ velocity (m/sec). To reinforce that interpretation, let's use t for x and v for f. Then $\int_a^b v(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i^*)\Delta t$.

Each $v(t_i^*)$ is a velocity (m/sec) and each Δt is a time (sec), so the units for each term $v(t_i^*)\Delta t$ are (meters/sec)(sec) $=$ meters. Therefore $\sum_{i=1}^{n} v(t_i^*)\Delta t$ is in meters and $\int_a^b v(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} v(t_i^*)\Delta t$ is in meters: it represents the net change of position between times $t = a$ and $t = b$.

Important properties of the integral and proofs of some of them

Suppose f, g are integrable on $[a,b]$, that is, suppose $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists.

(*This is true, for example, if f is continuous; and also true if f has only a finite number of jump discontinuities (discontinuities where $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but aren't equal.*) Then we say that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

<u>In every case, you should think what the property means in terms of areas. Then suppose x represents time (sec) and $f(x)$ represents velocity (m/sec), and decide what each property is saying.</u>

1. If c is a constant, $\int_a^b c\,dx = c(b-a)$

Why? In the definition, every $f(x_i^*) = c$, so

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} c\,\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x = c\lim_{n\to\infty}(b-a) = c(b-a).$$

2. $\int_a^b f(x)\pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Why? Let $h(x) = f(x)\pm g(x)$. Then

$$\int_a^b f(x)\pm g(x)\,dx = \int_a^b h(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} h(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}(f(x_i^*)\pm g(x_i^*))\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \pm \sum_{i=1}^{n} g(x_i^*)\Delta x = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

Why?   Let $h(x) = cf(x)$. Then

$$\int_a^b cf(x)\,dx = \int_a^b h(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} h(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i^*)\Delta x$$

$$= c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = c\int_a^b f(x)\,dx.$$

4. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

This one is a bit hard to prove in general, but thinking about areas makes it believable.

5. If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

Why? Since $f \geq 0$, then every $f(x_i^*) \geq 0$ in the definition. Therefore each $f(x_i^*)\Delta x \geq 0$, so $\sum_{i=1}^{n} f(x_i^*)\Delta x \geq 0$. Therefore $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \geq 0$.

6. If $f(x) \geq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Why? $f(x) - g(x) \geq 0$, so, using 5), $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)$ $dx \geq 0$, so $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

7. If m, M are constants and $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Why? In the definition, every $f(x_i^*) \leq M$, so $\sum_{i=1}^{n} f(x_i^*)\Delta x \leq \sum_{i=1}^{n} M\Delta x$, so $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \leq \lim_{n\to\infty} \sum_{i=1}^{n} M\Delta x = M \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x = M \lim_{n\to\infty} (b-a) = M(b-a)$.