Review of Inverses

(*See text material in §1.5 and 1.6*)

Functions f and g are called <u>inverses</u> to each other if each one "undoes" the effect of the other, that is : for a in the domain of f and b in the domain of g,

$$\begin{cases} 1)\ \text{if } f(a) = b,\ \text{then } g(b) = a & (\text{so that } g(f(a)) = a), \text{ and} \\ 2)\ \text{if } g(b) = a,\ \text{then } f(a) = b & (\text{so that } f(g(b)) = b) \end{cases}$$

Schematically, $a \overset{f}{\longrightarrow} b$

$a \overset{g}{\longleftarrow} b$

For example, $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ are inverses because

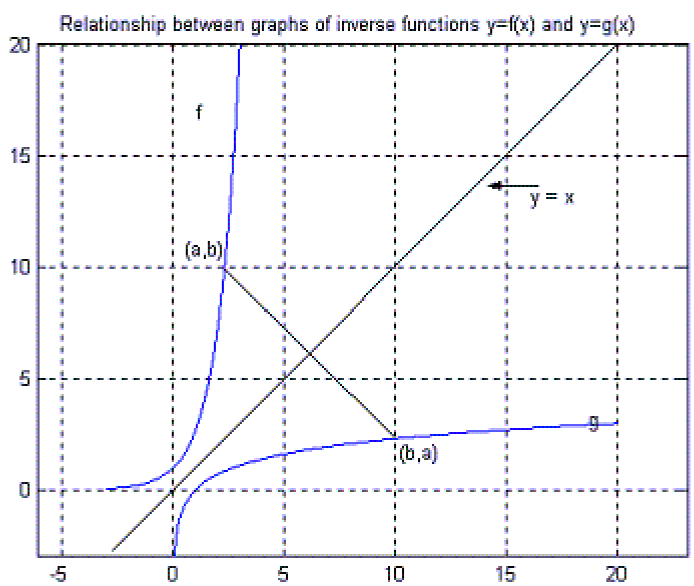
$$\begin{cases} 1)\ g(f(a)) = g(2a) = \frac{1}{2}(2a) = a, \quad and \\ 2)\ f(g(b)) = f(\frac{1}{2}b) = 2(\frac{1}{2}b) = b. \end{cases}$$

<u>For inverse functions</u> f and g:

domain f = range g, and
domain g = range f (from Equations 1) and 2))

(a, b) is on the graph of f if and only if $b = f(a)$
if and only if $a = g(b)$
if and only if (b, a) is on the graph of g

(*See the figure*)



The preceding figure shows how the points (a, b) and (b, a) are related: each is the reflection of the other across the line $y = x$. Therefore <u>the graphs of inverse functions f and g are reflections of each</u>

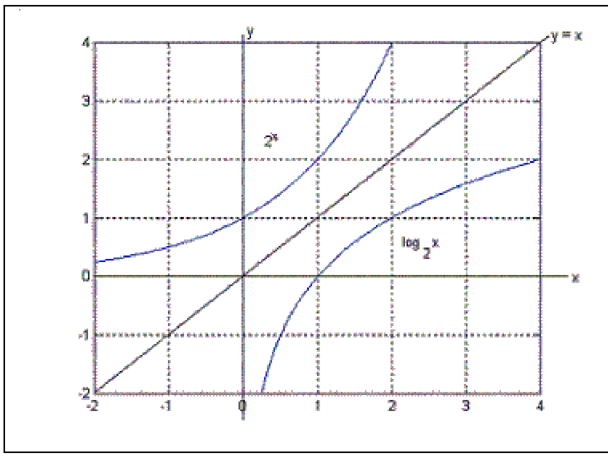
other across the line $y = x$. (*For example, carefully draw the graphs of* $f(x) = 2x$ *and* $g(x) = \frac{1}{2}x$ *to see a specific example of this reflection property of inverse functions.*)

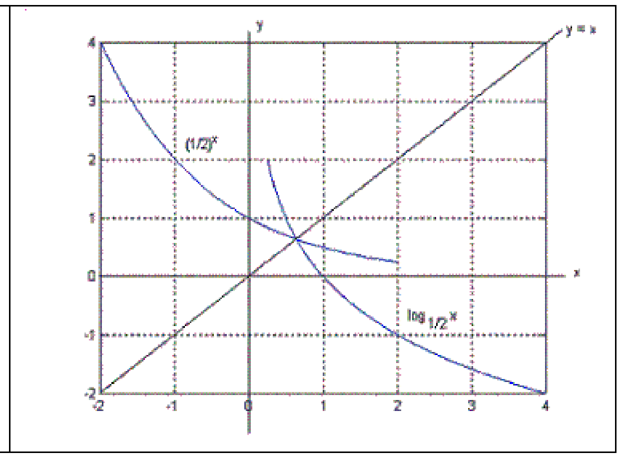
Review of Exponential and Logarithm functions

Exponential functions and logarithm functions are inverses of each other. The inverse of $f(x) = a^x$ is $g(x) = \log_a x$. (*Here,* $a > 0$ *and* $a \neq 1$. *Since* 1^x *is constant we don't want to call it an "exponential" function; and we want to avoid functions like* $(-2)^x$ *because they aren't defined for certain* x*'s – for example,* $(-2)^{1/2} =$?)

The domain of $f(x) = a^x$ is the set of all real numbers $(-\infty, \infty)$ and the range is the set of all positive real numbers $(0, \infty)$. Therefore the domain for the inverse function $g(x) = \log_a x$ is $(0, \infty)$ its range is $(-\infty, \infty)$. .

The two figures below show (on the left) the graphs of 2^x and $\log_2 x$ and (on the right) the graphs of $(\frac{1}{2})^x$ and $\log_{\frac{1}{2}} x$ (*but most often we will be using a base* $a > 1$). The graphs are reflections of each other across the line $y = x$.





Because $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions, Equations 1) and 2) tell us that $f(g(x)) = x$ and $g(f(x)) = x$, that is

$$(**) \quad a^{\log_a x} = x \qquad \text{for } x \text{ in } (0,\infty) \text{ and}$$
$$\log_a(a^x) = x \qquad \text{for } x \text{ in } (-\infty,\infty)$$

For example, $10^{\log_{10} 13} = 13$ and $\log_5(5^9) = 9$.

We can think of the equation $a^{\log_a x} = x$ as saying: " $\log_a x$ is the power (<u>exponent</u>) to which a must be raised to give x." If we put it that way,

$\log_{10} 100 = 2$ because 2 is the exponent to which 10 must be raised to give 100

$\log_3 81 = 4$ because 4 is the exponent to which 3 must be raised to give 81.

$\log_a 1 = 0$ because $a^0 = 1$.
Therefore $(1,0)$ *is on the graph of* $y = \log_a x$ *for any base* a; *we should have known this anyway – because* $(0,1)$ *is on the graph of* $y = a^x$.

Since logarithms can be thought of as <u>exponents</u>, they have properties like exponents. For example:

$\log_a(xy) = ?$

This is asking "$a^? = xy$".
The answer is $\log_a x + \log_a y$, <u>because</u> the laws of exponents say

$$a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = xy$$

Therefore

$$\boldsymbol{\log_a xy = \log_a x + \log_a y}$$

$\log_a(x^r) = ?$

This is asking "$a^? = x^r$."
The answer is $r \log_a x$ because the laws of exponents say $a^{r\log_a x} = (a^{\log_a x})^r = x^r$.

Therefore

$$\boldsymbol{\log_a(x^r) = r \log_a x}$$

Examples:

1) $\log_3(48) = \log_3(3 \cdot 16) = \log_3 3 + \log_3(16) = 1 + \log_3 16$

2) $\log_{10}(400) = \log_{10}(10^2) + \log_{10}(4) = 2 + \log_{10} 4$

3) To solve $10^{3x^2-4} = 7$, we can "kill" the exponential with a logarithm:

$$\log_{10}(10^{3x^2-4}) = \log_{10} 7$$
$$3x^2 - 4 = \log_{10} 7$$
$$x^2 = (4 + \log_{10} 7)/3$$
$$x = \pm\sqrt{(4 + \log_{10} 7)/3}$$

4) To solve $\log_9(7x-3) = 10$, we can "kill" the logarithm with an exponential:

$$9^{\log_9(7x-3)} = 9^{10}$$
$$7x - 3 = 9^{10}$$
$$x = (9^{10} + 3)/7$$

The base most often used in calculus is base $a = e$, because (as we have seen) the exponential $f(x) = a^x$ has the simplest derivative when $a = e$:

$\frac{d}{dx}(a^x) = (\ln a) \cdot a^x$, and, when $a = e$,

$\frac{d}{dx}(e^x) = (\ln e) \cdot e^x = 1 \cdot e^x = e^x$

For $y = f(x) = e^x,$ the tangent line at $(0, 1)$ on the graph has slope $f'(1) = 1.$
For $y = f(x) = a^x,$ the tangent line at $(0, 1)$ on the graph has slope $f'(1) = \ln a.$

As we know, e is a constant whose value is approximately 2.71828. The exponential function $y = e^x$ has $y = \log_e x$ as its inverse. Since we most often use base e, $\log_e x$ is given a simpler name: $\log_e x = \ln x$.

We can use properties of logarithms to see how to convert logs from one base to another.

The conversion formula is:

$$(***) \quad \boldsymbol{\log_b x = \frac{\log_a x}{\log_a b}}$$

(In remembering this: imagine you already know how to compute logs with base a and want to do logs in base b instead: the "new thing", $\log_b x$ (on the left), is given in terms of the "old things": all the logs on the right side of the equation are base a.

Why is (***) true?

Equation (***) is the same as saying:

$$(\log_b x)(\log_a b) = \log_a x$$

To verify that, we need to check whether

$$a^{(\log_b x)(\log_a b)} = x$$

But that's true because of laws of exponents:

$$a^{(\log_b x)(\log_a b)} = ((a^{\log_a b})^{(\log_b x)}$$

$$= b^{\log_b x} = x.$$

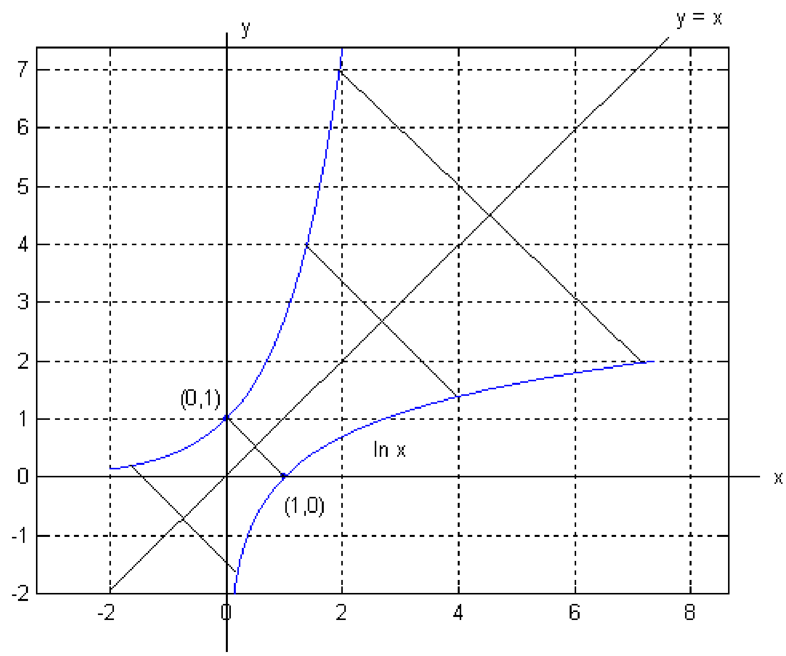
For example,

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$

Therefore if your calculator can do "ln", you can also use it to find the value of, say, $\log_{17} 35$.

For base $a = e$, the earlier equations (**) become: $e^{\ln x} = x$ and $\ln(e^x) = x$

The graphs look like:



Notice how in reflecting across the line $y = x$, the horizontal asymptote (x-axis) of $y = e^x$ turns into a vertical asymptote (the y-axis) for the graph of the inverse $y = \ln x$.

We know that for $f(x) = e^x$, $f'(x) = f''(x) = e^x$. Since the first and second derivatives are both always > 0, e^x is always increasing and concave up (*as the graph seems to show.*).

We will see soon that for $g(x) = \ln x$:

$$g'(x) = \frac{1}{x}, \qquad \text{and so} \qquad g''(x) = -\frac{1}{x^2}$$

Since $g(x)$ is only defined for $x > 0$, $\frac{1}{x}$ is always > 0 so the graph of $y = \ln x$ is always increasing. Since $-\frac{1}{x^2} < 0$ always, the graph of $y = \ln x$ is always concave down (*as the graph seems to indicate*).

Notice finally that:

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0$$
$$\lim_{x\to 0^+} \ln x = \infty \text{ and } \lim_{x\to\infty} \ln x = \infty.$$

The last limit seems a little counterintuitive: $(\ln x)' = \frac{1}{x} \to 0$ as $x \to \infty$: the graph "flattens out," so one might feel like there's a horizontal asymptote. But a little thought shows that this is false: $\ln x$ can be made as large as we like by taking x large enough. For example: $\ln x > 1000$ if $e^{\ln x} > e^{1000}$. Of course, e^{1000} is very large: $e^{1000} > 10^{333}$ (why?!?). It takes a long while for the graph of $\ln x$ to rise above height 1000, but it does eventually do so.