Sums of Powers of Natural Numbers

We'll use the symbol S_k for the sum of the k^{th} powers of the first n natural numbers. In other words,

$$S_k = 1^k + 2^k + \ ... \ + n^k.$$

Of course, this is a "formula" for S_k, but it doesn't help you compute – it doesn't tell you how to find the exact value, say, of $S_3 = 1^3 + 2^3 + ... + 15^3$. We'd like to get what's called a closed formula for S_k, that is, one without the annoying " ... " in it.

For $S_0 = 1^0 + 2^0 + \ ... \ + n^0$, this is easy: since there are n terms, each equal to 1, so we get

$$S_0 = 1 + 1 + ... + 1 \ = 1 \cdot n \ = n$$

For S_1, it's already harder. Here's a slick way of finding a closed formula for S_1:

Write down S_1 twice, in two different orders:

$$\begin{array}{l} S_1 = \quad 1 \quad + \quad 2 \quad + \quad 3 \quad + \ ... \ + \ (n-1) \ + \quad n, \qquad \text{and} \\ S_1 = \quad n \quad + \ (n-1) \ + \ (n-2) \ + ... \ + \quad 2 \quad + \quad 1 \qquad \text{Then add to get:} \\ \hline 2S_1 = (n+1) \ + \ (n+1) \ + \ (n+1) \ + ... \ + \ (n+1) \ + \ (n+1). \end{array}$$

Since there are n terms on the right, each equal to $(n+1)$, we get

$2S_1 = n(n+1)$, so

$$S_1 \ = \frac{n(n+1)}{2}$$

This is a "usable" closed formula: for example, $1 + 2 + 3 + ... + 15 \ = \frac{15(16)}{2} = 120$.

Here's a list of formulas for:

$$S_0 = 1^0 + 2^0 + \ ... \ + \ n^0 = n$$

$$S_1 = 1^1 + 2^1 + ... \ + n^1 = \frac{n(n+1)}{2}$$

$$S_2 = 1^2 + 2^2 + ... \ + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = [\frac{n(n+1)}{2}]^2 \qquad \text{(Curious observation: } S_3 = [S_1]^2\text{)}$$

Where do these formulas come from? There is a systematic way to get a formula for each S_k once you know the previous formulas for $S_0, S_1, \ldots, S_{k-1}$: We'll illustrate here with just two examples:

i) If someone noticed the (easy) fact the $S_0 = 1^0 + 2^0 + \ldots + n^0 = 1 + 1 + \ldots + 1 = n$
how could this fact be used to get a formula: $S_1 = 1^1 + 2^1 + \ldots + n^1 = 1 + 2 + \ldots + n =$???

Well, for any positive integer j, we know that $(j+1)^2 - j^2 = 2j + 1$. So we write this down substituting in each value $j = 1,\ j = 2,\ \ldots,\ j = n$

$$\begin{aligned}
2^2 - 1^2 &= 2(1) + 1 \\
3^2 - 2^2 &= 2(2) + 1 \\
4^2 - 3^2 &= 2(3) + 1 \\
&\vdots \\
(n+1)^2 - n^2 &= 2(n) + 1
\end{aligned}$$

Adding up the columns on both sides (with lots of cancellations on the left-hand side) gives

$$\begin{aligned}
(n+1)^2 - 1 &= 2(1 + 2 + \ldots + n) + n \\
&= 2 \quad S_1 \quad + n \quad \text{so} \\
n^2 + 2n + 1 - 1 &= 2S_1 + n \quad \text{so} \\
n^2 + n &= 2S_1 \quad \text{so} \\
\frac{n^2+n}{2} = \frac{n(n+1)}{2} &= S_1.
\end{aligned}$$

ii) So now we know formulas for S_0 and S_1. How can use these to we get a formula for $S_2 = 1^2 + 2^2 + \ldots + n^2$= ??? It's the same idea, but just a little more algebra.

For any j we know that $(j+1)^3 - j^3 = 3j^2 + 3j + 1$. So we write this down substituting each value $j = 1,\ j = 2,\ \ldots,\ j = n$

$$\begin{aligned}
2^3 - 1^3 &= 3(1^2) + 3(1) + 1 \\
3^3 - 2^3 &= 3(2^2) + 3(2) + 1 \\
4^3 - 3^3 &= 3(3^2) + 3(3) + 1 \\
&\vdots \\
(n+1)^3 - n^3 &= 3(n^2) + 3(n) + 1.
\end{aligned}$$

Adding up the columns on both sides (with lots of cancellations on the left-hand side) gives

$$\begin{aligned}
(n+1)^3 - 1 &= 3\,(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + \ldots + n) + (1 + 1 + .. + 1) \\
&\qquad\quad \downarrow \qquad\qquad\qquad\qquad\quad \downarrow \qquad\qquad\qquad \downarrow \\
&= 3 \qquad S_2 \qquad\qquad + 3 \qquad S_1 \qquad + \qquad\quad S_0\,,
\end{aligned}$$

so

$n^3 + 3n^2 + 3n + 1 - 1 = 3S_2 + 3S_1 + S_0$. Then we solve to get S_2.

$n^3 + 3n^2 + 3n - 3S_1 - S_0 = 3S_2$, so
$S_2 = \frac{n^3+3n^2+3n-3S_1-S_0}{3}$.

We can substitute the formulas we already know for S_0 and S_1 and simplfy to ge

$$S_2 = \frac{n^3 + 3n^2 + 3n - 3[\frac{n(n+1)}{2}] - n}{3} = \ldots = \ = \frac{n(n+1)\,(2n+1)}{6}$$

Optional Material

If you know the binomial formula (from high school) and can therefore expand $(j+1)^k$, then the same idea works for any natural number k. But the bigger k is, the more lagebra is involved. An outline goes like this.

The formula for the "binomial coefficients" : $\binom{k}{l} = \frac{k!}{l!\,(k-l)!}$)

Suppose we have figured out formulas for $S_0, S_1, S_2, \ldots, S_{k-1}$. We know (from the binomial theorem) that for any j,

$$(j+1)^{k+1} - j^{k+1} = \binom{k+1}{1} j^k + \binom{k+1}{2} j^{k-1} + \binom{k+1}{3} j^{k-2} + \ldots + 1$$

Write this out for each value $j = 1, 2, \ldots n$.

$$\begin{aligned}
2^{k+1} \quad - 1^{k+1} &= \binom{k+1}{1} 1^k + \binom{k+1}{2} 1^{k-1} + \binom{k+1}{3} 1^{k-2} \ldots + 1 \\
3^{k+1} \quad - 2^{k+1} &= \binom{k+1}{1} 2^k + \binom{k+1}{2} 2^{k-1} + \binom{k+1}{3} 2^{k-2} \ldots + 1 \\
\ldots\ldots & \\
(n+1)^{k+1} - n^{k+1} &= \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-1} + \binom{k+1}{3} n^{k-2} \ldots + 1.
\end{aligned}$$

<u>Add the columns:</u>

$$\begin{aligned}
(n+1)^{k+1} - 1 &= \binom{k+1}{1}(1^k + 2^k + \ldots + n^k) + \binom{k+1}{2}(1^{k-1} + 2^{k-1} + \ldots + n^{k-1}) \\
&\quad + \binom{k+1}{3}(1^{k-2} + 2^{k-2} + \ldots + n^{k-1}) \quad \ldots + n \\
&= \binom{k+1}{1} S_k + \binom{k+1}{2} S_{k-1} + \binom{k+1}{3} S_{k-2} + \ \ldots\ + S_0.
\end{aligned}$$

Then we solve for what we want:

$$\begin{aligned}
S_k &= [(n+1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ \ldots\ - S_0] / \binom{k+1}{1} \\
&= [(n+1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ \ldots\ - S_0] / (k+1)
\end{aligned}$$

We are assuming that we already have formulas for $S_{k-1}, S_{k-2}, \ldots S_1, S_0$ $-$ which we then substitute into this formula to get one closed, if complicated, formula for S_k in terms of n. Try it to find a formula for

$$S_4 = 1^4 + 2^4 + \ldots + n^4 = \ldots$$