TRIGONOMETRIC IDENTITIES

- **Reciprocal identities**

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u}$$
$$\tan u = \frac{1}{\cot u} \quad \cot u = \frac{1}{\tan u}$$
$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u}$$

- **Pythagorean Identities**

$$\sin^2 u + \cos^2 u = 1$$
$$1 + \tan^2 u = \sec^2 u$$
$$1 + \cot^2 u = \csc^2 u$$

- **Quotient Identities**

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$

- **Co-Function Identities**

$$\sin(\frac{\pi}{2} - u) = \cos u \quad \cos(\frac{\pi}{2} - u) = \sin u$$
$$\tan(\frac{\pi}{2} - u) = \cot u \quad \cot(\frac{\pi}{2} - u) = \tan u$$
$$\csc(\frac{\pi}{2} - u) = \sec u \quad \sec(\frac{\pi}{2} - u) = \csc u$$

- **Parity Identities (Even & Odd)**

$$\sin(-u) = -\sin u \quad \cos(-u) = \cos u$$
$$\tan(-u) = -\tan u \quad \cot(-u) = -\cot u$$
$$\csc(-u) = -\csc u \quad \sec(-u) = \sec u$$

- **Sum & Difference Formulas**

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

- **Double Angle Formulas**

$$\begin{aligned}
\sin(2u) &= 2\sin u \cos u \\
\cos(2u) &= \cos^2 u - \sin^2 u \\
&= 2\cos^2 u - 1 \\
&= 1 - 2\sin^2 u \\
\tan(2u) &= \frac{2\tan u}{1 - \tan^2 u}
\end{aligned}$$

- **Power-Reducing/Half Angle Formulas**

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$
$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$
$$\tan^2 u = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

- **Sum-to-Product Formulas**

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

- **Product-to-Sum Formulas**

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

Q1 A substitution turns $\int \tan^4 x \sec^4 x \, dx$ into $\int P(u)\, du$, where $P(u)$ is a polynomial. What is $P(u)$?

A) $2u^4$ B) $u^4 + u^2$ C) $u^3 + u^2$ **D)** $\boldsymbol{u^4 + u^6}$ E) $u^4 - u^6$

Answer $\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dx$ (let $u = \tan x$)

$$= \int u^4 (1+u^2)\, du = \int u^4 + u^6 \, du = \ldots$$

Q2 A substitution turns $\int \tan^3 x \sec^3 x \, dx$ into $\int P(u)\, du$, where $P(u)$ is a polynomial. What is $P(u)$?

A) $2u^4$ **B)** $\boldsymbol{u^4 - u^2}$ C) $u^3 + u^2$ D) $u^4 + u^6$ E) $u^4 - u^6$

Answer $\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx$ (let $u = \sec x$)

$$= \int (u^2 - 1)u^2 \, du = \int u^4 - u^2 \, du = \ldots$$

The example $\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$ was done in class but it is also done in the book and not reproduced here.

Example $\int \tan^2 x \sec x \, dx$ does not fit either a) or b) in the strategy outlined above. There is no systematic way to handle this case, but it can sometimes help to convert, if possible into other trig functions:

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1)\sec x \, dx = \int \sec^3 x - \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx = \quad \textit{(from earlier examples)}$$

$$= \tfrac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$$

$$= \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln|\sec x + \tan x| + C$$

Trigonometric Substitution

The formula of "u-substitution" that we have been using read like

$(*)\qquad \int f(g(x))g'(x)dx \quad (\text{let } u = g(x),\ du = g'(x)\,dx) \ = \int f(u)\,du$

Notice that when you use this formula in practice, you

i) choose a (hopefully helpful) substitution $\boldsymbol{u = g(x)}$,

ii) work out $\int f(u)du =$ the antiderivatives in terms of u and

iii) finally, substitute $u = g(x)$ to get back to the original variable x.

For example, $\int 2x\cos x^2\,dx$

i) let $u = x^2$, $du = 2x\,dx$ so $\int 2x \cos x^2\,dx = \int \cos u\,du$

ii) $= \sin u + C$

iii) use $u = x^2$ to get back to original variable x:

$\sin u + C = \sin x^2 + c$

The particular letters used don't matter. In (*) if we replace u by x and x by t (and exchange the left/right sides of the equation) we get

(**) $\quad \int f(x)\,dx = \int f(g(t))g'(t)dt$

This is really, the same equation but psychologically it suggests a different point of view. Starting with $\int f(x)\,dx$ we can substitute $\boldsymbol{x = g(t)}$, $dx = g'(t)\,dt$ to get $\int f(g(t))g'(t)dt$.

Notice that when you use this formula in practice, you

i) choose a (hopefully helpful) substitution $\boldsymbol{x = g(t)}$

ii) work out $\int f(g(t))g'(t)dt =$ the antiderivatives in terms of t and

iii) finally, **substitute** $\boldsymbol{t = ???}$ to get the antiderivative back into the original variable x.

↑

we need a formula $\boldsymbol{t = h(x)}$ to do this $-$ in other words we have to be able to solve $x = g(t)$ for t, getting $t = h(x)$ (h is the inverse function to the function g. For example, if $x = \ln t$, then $t = e^x$.

This new sort of substitution come up in the topic of trigonometric substitutions.

Here is one kind of trigonometric substitution

When $\sqrt{a^2 - x^2}$
occurs in a integral try the substitution
(we can assume $a > 0$)

$$x = a\sin\theta$$
$$dx = a cos\theta\, d\theta$$

(*the inverse function we will need later is then*

$$\theta = \text{arcsin}\left(\tfrac{x}{a}\right)$$

(where, as usual, arcsin always gives $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$)

is sometimes useful because then

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta} = \sqrt{a^2\cos^2\theta}$$
$$= a\cos\theta.$$
$$\uparrow$$

(*since* $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we know $\cos\theta \ge 0$, so $\sqrt{cos^2\theta} = |cos\,\theta| = \cos\theta$; *and since we assume* $a > 0$, $\sqrt{a^2} = |a| = a$)

Example $\int \frac{1}{\sqrt{8 - x^2}}\, dx$ Here, $a^2 = 8$, so $a = \sqrt{8}$.

Substitute
$$x = \sqrt{8}\sin\theta$$
$$dx = \sqrt{8}\cos\theta\, d\theta$$

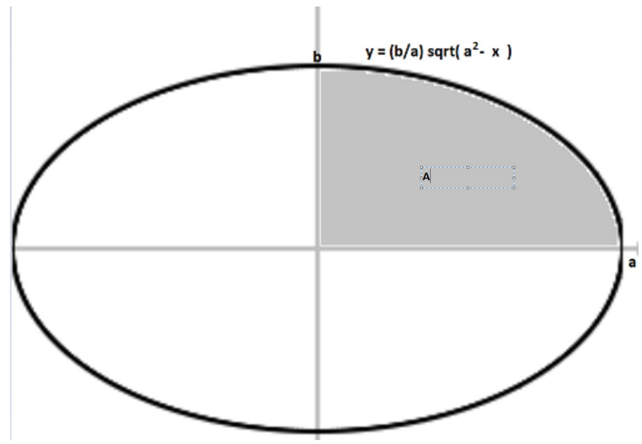
The inverse function we will need is $\theta = \text{arcsin}(\frac{x}{\sqrt{8}})$

Then $\int \frac{1}{\sqrt{8 - x^2}}\, dx = \int \frac{1}{\sqrt{8 - 8\sin^2\theta}}\sqrt{8}\cos\theta\, d\theta = \int \frac{1}{\sqrt{8\cos^2\theta}}\sqrt{8}\cos\theta\, d\theta =$

$$\int \frac{1}{\sqrt{8}\cos\theta}\sqrt{8}\cos\theta\, d\theta = \int 1\, d\theta = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\text{arcsin}(\tfrac{x}{\sqrt{8}}) + C$$
$$\uparrow$$

here is where the inverse function for the substitution is needed

Example Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The total area within the ellipse (by symmetry) is $4A$, where A is the area shaded.

Since $b^2x^2 + a^2y^2 = a^2b^2$

$$a^2y^2 = a^2b^2 - b^2x^2 = b^2(a^2 - x^2)$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$ is the equation for the top boundary curve of the shaded region

So total area within ellipse

$$= 4A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \quad (let\, x = a\ sin\ \theta,\ dx = a\ cos\ \theta)\ d\theta)$$

$$when\ x = a,\ \theta = \tfrac{\pi}{2};\ when\ x = 0,\ \theta = 0)$$

$$= 4\frac{b}{a}\int_0^{\pi/2}\sqrt{a^2 - a^2\sin^2\theta}\,a\cos\theta\,d\theta = 4\frac{b}{a}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta$$

$$= 4\frac{b}{a}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 4ab\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = 4ab(\,(\tfrac{\pi}{4} - 0) - (0 - 0)) = \pi ab.$$

So the area of an ellipse is π(length of semimajor axis)(length of semiminor axis) $= \pi ab$.

Notice that if $a = b$, then the ellipse is actually a circle $x^2 + y^2 = a^2$, and it encloses the an area $\pi ab = \pi a^2$, the usual formula for the area of a circle !