Q1 If $a^2 - x^2$ or $\sqrt{a^2 - x^2}$ occurs as part of an integral, then which substitution might be useful ?

A) $x = a^2 \sin\theta$ B) $x = a^2 \sec\ \theta$ C) $x = a \tan\ \theta$
D) $x = a \sin\theta$ E) $x = a \sec\ \theta$

The substitution to try (particularly for $\sqrt{a^2 - x^2}$) is $x = a\sin\theta$

because then $a^2 - x^2 \ = a^2 - a^2 \sin\theta = a^2 \cos\theta$ and $\sqrt{x^2 - a^2}$

$= \sqrt{a^2 \cos^2\theta} \ = a\cos\theta$

In the case of $\sqrt{a^2 - x^2}$, this gets the $\sqrt{\ }$ out of the integral (*at the cost of introducing trig functions into the problem*)

Q2 If $x^2 - a^2$ or $\sqrt{x^2 - a^2}$ occurs as part of an integral, then which substitution might be useful ?

A) $x = a^2 \sin\theta$ B) $x = a^2 \sec\ \theta$ C) $x = a \tan\ \theta$
D) $x = a \sin\theta$ E) $x = a \sec\ \theta$

The substitution to try (particularly for $\sqrt{x^2 - a^2}$) is $x = a\sec\theta$

because then $x^2 - a^2 \ = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2 \tan^2\theta,$

and $\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2\theta} = a\tan\theta$

Q3 If $x^2 + a^2$ or $\sqrt{x^2 + a^2}$ occurs as part of an integral, then which substitution might be useful ?

A) $x = a^2 \sin\theta$ B) $x = a^2 \sec\ \theta$ C) $x = a \tan\ \theta$
D) $x = a \sin\theta$ E) $x = a \sec\ \theta$

The substitution to try (particularly for $\sqrt{x^2 + a^2}$) is $x = a\tan\theta$

because then $x^2 + a^2 \ = a^2 \tan^2\theta + a^2 = a^2(\tan^2\theta + 1) = a^2 \sec^2\theta,$

and $\sqrt{x^2 + a^2} = \sqrt{a^2 \sec^2\theta} = a\sec\theta$

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

The lecture contained 3 examples of trigonometric substitutions.

Example In this example, the choice of substitution is straightforward. But trig identities are used along the way, and some simplification of an expression involving an inverse trig function occurs at the end.

$$\int \frac{x^2}{\sqrt{9-x^2}} dx = \qquad \text{(Let } x = 3\sin\theta,\ dx = 3\cos\theta\, d\theta\text{)}$$

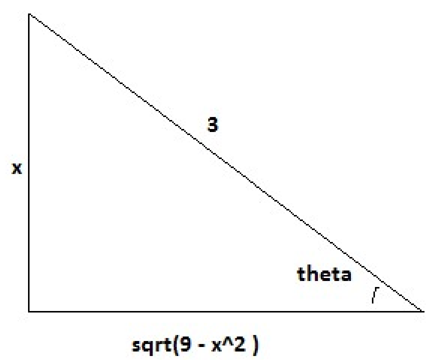
$$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\ \theta\, d\,\theta = \int \frac{9\sin^2\theta}{2\cos\theta} 3\cos\theta\, d\theta = \int 9\sin^2\theta\, d\theta$$

$\downarrow$ *trig identity* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$ *trig identity*

$$= \frac{9}{2}\int 1 - \cos 2\theta\ \ d\theta = \frac{9}{2}(\theta - \frac{1}{2}\sin 2\theta) + C = \frac{9}{2}(\theta - \frac{1}{2}\, 2\sin\theta\cos\theta) + C$$

$\downarrow$ *since* $\theta = arctan(\frac{x}{3})$

$$= \frac{9}{2}(\arctan(\frac{x}{3}) - \mathbf{\sin(\arctan\ \frac{x}{3})\ \cos(\arctan\ \frac{x}{3})}$$



$$= \frac{9}{2}(\arctan(\frac{x}{3}) - \frac{x}{3}\ (\frac{\sqrt{9-x^2}}{3}) = \frac{9}{2}\arctan\,(\frac{x}{3}) - \frac{x}{2}\sqrt{9-x^2} + C$$

The next example show how “completing a square” might help.

<u>Example</u> $\int \frac{1}{2x^2+8x+12}\,dx$

There is nothing of the form $x^2+a^2,\ a^2-x^2$, or x^2-a^2 in sight. But we can complete a square to get into a form where a trig substitution appears useful.

Notice $2x^2\ +\ 8x+12$

$$\begin{aligned} &= 2(x^2+4x\qquad)+12 \\ &= 2(x^2+4x+4)+12-8 \\ &= 2(x+2)^2+4 \end{aligned}$$

So $\int \frac{1}{2x^2+8x+12}\,dx \quad = \int \frac{1}{2(x+2)^2+4}\,dx = \frac{1}{2}\int \frac{1}{(x+2)^2+2}dx$ (let $u = x+2,\ du = dx$)

$= \frac{1}{2}\int \frac{1}{u^2+2}du$ (let $u = \sqrt{2}\tan\theta,\ du = \sqrt{2}\sec^2\theta\,d\theta$)

$= \frac{1}{2}\int \frac{1}{2\tan^2\theta+2}\sqrt{2}\sec^2\theta\,d\theta = \frac{\sqrt{2}}{4}\int \frac{1}{\tan^2\theta+1}\sec^2\theta\,d\theta = \frac{\sqrt{2}}{4}\int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta$

$= \frac{\sqrt{2}}{4}\int 1\ d\theta = \frac{\sqrt{2}}{4}\,\theta + C = \frac{\sqrt{2}}{4}\ \text{arctan}(\frac{u}{\sqrt{2}}) + C = \frac{\sqrt{2}}{4}\ \text{arctan}(\frac{x+2}{\sqrt{2}}) + C$

This example illustrates an $x = a\sec\theta$ substitution

<u>Example</u> $\int \frac{dx}{x^4\sqrt{x^2-2}}$ let $x = \sqrt{2}\sec\theta$ $dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$

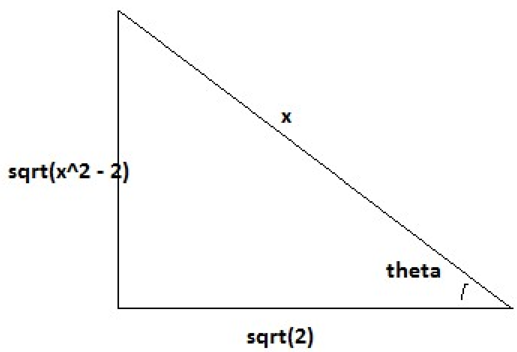
$\theta = \text{arcsec}(\frac{x}{\sqrt{2}})$

$$= \int \frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{4\sec^4\theta\sqrt{2\sec^2\theta - 2}} = \int \frac{\sec\theta\tan\theta\ d\theta}{4\sec^4\theta\ \tan\theta} = \frac{1}{4}\int \cos^3\theta\, d\theta$$

$$= \frac{1}{4}\int \cos^2\theta\cos\theta\, d\theta = \frac{1}{4}\int(1-\sin^2\theta)\cos\theta\, d\theta$$ let $u = \sin\theta$

$$= \frac{1}{4}\int(1-u^2)\ du = \frac{u}{4} - \frac{u^3}{12} = \frac{1}{4}(\sin\theta - \frac{1}{3}\sin^3\theta) + C$$

$$= \frac{1}{4}(\sin(\text{arcsec}(\frac{x}{\sqrt{2}}) - \frac{1}{3}(\sin^3(\text{arcsec}(\frac{x}{\sqrt{2}})) + C$$



$$= \frac{1}{4}(\frac{(x^2-2)^{1/2}}{x} - \frac{(x^2-2)^{3/2}}{3x^3}) + C$$

$$= \frac{1}{4}(x^2-2)^{1/2}\,(\frac{1}{x} - \frac{x^2-2}{3x^3})\ + C$$

$$= \frac{1}{4x}(x^2-2)^{1/2}\,(1 - \frac{x^2-2}{3x^2})\ + C$$

$$= \frac{1}{4x}(x^2-2)^{1/2}\,(\frac{2x^2+2}{3x^2})\ + C$$

$$= \frac{1}{2x}(x^2-2)^{1/2}\,(\frac{x^2+1}{3x^2})\ + C$$

These examples were "leftover" in my notes and not done in class. The first also involves completing a square.

<u>Example</u> $\int_0^1 \sqrt{x - x^2}dx = \int_0^1 \sqrt{-(x^2 - x)}\, dx = \int_0^1 \sqrt{-(x^2 - x + \frac{1}{4}) + \frac{1}{4}}\, dx$

$$= \int_0^1 \sqrt{\tfrac{1}{4} - (x - \tfrac{1}{2})^2}\, dx \qquad \text{Let } u = x - \tfrac{1}{2}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\tfrac{1}{4} - u^2}\, du \qquad \text{Let } u = \tfrac{1}{2}\sin\theta$$

$$du = \tfrac{1}{2}\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\ \tfrac{1}{2}\cos\theta\, d\theta = 2\int_0^{\pi/2} \tfrac{1}{4}\cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1+\cos 2\theta}{2} d\theta = \tfrac{1}{4}(\theta + \tfrac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \tfrac{1}{4}(\tfrac{\pi}{2}) = \tfrac{\pi}{8}.$$

Notice that we saved some work because we converted the limits into "θ limits" and did not need to get the antiderivative back in terms of x! We also saved work by noticing that $\sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}\ \frac{1}{2}\cos\theta$ is an <u>even</u> function – so that we could switch to $2\int_0^{\pi/2} ***$.

To return to x and just get a general formula for the antiderivative

$$\int \sqrt{x - x^2}dx = \ldots.$$

$$= \int \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\ \tfrac{1}{2}\cos\theta\, d\theta = \tfrac{1}{4}\int \cos^2\theta\, d\theta = \tfrac{1}{4}\int \tfrac{1+\cos 2\theta}{2}\, d\theta$$

$$= \tfrac{1}{8}\,(\theta + \tfrac{1}{2}\sin 2\theta) = \tfrac{1}{8}(\theta + \sin\theta\cos\theta) = \tfrac{1}{8}(\arcsin 2u + \sin(\arcsin 2u)\cos(\arcsin 2u))$$

$$= \tfrac{1}{8}(\arcsin 2u + 2u\sqrt{1 - 4u^2})$$

$$= \tfrac{1}{8}\arcsin(2x - 1) + (2x - 1)\sqrt{1 - 4(2x - 1)^2} + C$$

<u>Example</u> $\int \frac{x^3}{4+x^2}\,dx$ let $x = 2\tan\theta,\ dx = 2\sec^2 \mathrm{d}\theta$

$$= \int \frac{8\tan^3\theta}{4+4\tan^2\theta}\, 2\sec^2\theta\, d\theta = 4\int \frac{\tan^3\theta}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$= 4\int \tan\theta\,(\sec^2\theta - 1)\, d\theta = 4\Big(\int \tan\theta \sec^2\theta\ d\theta - \int \tan\theta\, d\theta\Big)$$

$$= 4\,(\,\frac{\tan^2\theta}{2} - \ln|\sec\theta|\,) + C = 4(\frac{x^2}{8} - \ln|\frac{\sqrt{4+x^2}}{2}|\ + C$$

$$= \frac{x^2}{2} - 4\ln|\,\frac{\sqrt{4+x^2}}{2}| + C$$

(*Alternately, substitute* $u = x^2,\ \frac{1}{2}du = xdx$)

$$\int \frac{x^3}{4+x^2}\,dx = \int \frac{x^2\cdot x}{4+x^2}\,dx = \frac{1}{2}\int \frac{u}{4+u}\,du = \frac{1}{2}\int 1 - \frac{4}{4+u}\,du$$

$$= \frac{1}{2}(u - 4\ln|u+4| + C = \frac{x^2}{2} - 2\ln(x^2+4) + C$$