Q1: What trig substitution would be useful for $\int \frac{x}{8+2(x-1)^2}\, dx$?

A) $x = \sqrt{8}\sin\theta$ B) $x = \sqrt{2}\tan\theta$ C) $x = \sqrt{8}\,\tan\theta$

D) $x = 1 + 2\tan\theta$ E) $x = 1 - 2\sin\theta$

<u>Answer</u> $\int \frac{x}{8+2(x-1)^2}\, dx = \int \frac{x}{2(4+(x-1)^2)}\, dx$ (*let* $x - 1 = u,\ dx = du$)

$= \int \frac{u+1}{2(4+u^2)}\, du.$ Then let $u = 2\tan\theta$, <u>so that</u> $4 + u^2 = 4(1+\tan^2\theta) = 4\sec^2\theta$

So $x - 1 = u = 2\tan\theta$, or $x = 1 + 2\tan\theta$.

(*If you practice with these substitutions, you might do the* u*-substitution in your head and just directly let* $x = 1 + 2\tan\theta$.)

Partial Fractions

Suppose $P(x)$ and $Q(x)$ are polynomials; $\frac{P(x)}{Q(x)}$ is called a rational function.
We want to look as a method to find $\int \frac{P(x)}{Q(x)} dx$.

a) Mathematically, the method always works, but

b) the algebra involved (not the calculus) may make a problem too difficult without some computer assistance. We will look at relatively simple examples of the method.

The method is called partial fractions because we will write

$\frac{P(x)}{Q(x)} = \text{—} + \text{—} + \ldots + \text{—}$ = a sum of simpler fractions (simpler parts) that we already know how to integrate.

This sum of fractions will be called the partial fraction decomposition of $\frac{P(x)}{Q(x)}$, and

$$\int \frac{P(x)}{Q(x)} dx = \int \text{—}\, dx + \int \text{—}\, dx + \ldots + \int \text{—}\, dx$$

The goal, of course, is that the integrals on the right will be easier to do.

STEP 0) The preliminary thing required is that we work with a rational function where the degree of the numerator is less than the degree of the denominator. If that's not true at the beginning, we do a long division of polynomials to make it so;

Example $\frac{2x^3 - 3x^2 - 8x - 2}{x^2 - 2x - 3}$ where degree of numerator (3) is large than degree of denominator.

			$2x$	$+1$		
x^2	$-2x$	-3	$2x^3$	$-3x^2$	$-8x$	-2
			$2x^3$	$-4x^2$	$-6x$	
				x^2	$-2x$	-2
				x^2	$-2x$	-3
						1

The division stops when the degree of the remainder is less than the degree of the divisor $x^2 - 2x - 3$

Then $\frac{2x^3-3x^2-8x-2}{x^2-2x-3}$ = quotient + $\frac{\text{remainder}}{\text{divisor}}$ $= \quad 2x+1+ \quad \frac{1}{x^2-2x-3}$

So $\int \frac{2x^3-3x^2-8x-2}{x^2-2x-3} \quad dx = \int 2x+1\, dx + \int \frac{1}{x^2-2x-3}\, dx$

$\uparrow$ easy: a polynomial

$\uparrow$ degree of numerator (1) $<$ degree of denominator (2)

Q2: $\int \frac{2x^3+2x^2+1}{x+1}\, dx = \int$ (polynomial $S(x)$) $+ (\frac{R}{x+1})\, dx$ where R is a constant.

What is R?

A) 0 B) 1 C) 2 D) -1 E) -2

Answer The long division gives a remainder $R=1$; in fact

$$\frac{2x^3+2x^2+1}{x+1} = 2x^2 + \frac{1}{x+1}$$

STEP 1) Completely factor the denominator $Q(x)$.

Doing this might be difficult, in practice. But in theory, it is a consequence a theorem called The Fundamental Theorem of Algebra (*often proved in Math 430*) that $Q(x)$ can always be factored (using only real numbers) into a product of linear factors (like, say, $2x-5$) and irreducible quadratic factors (like, say. x^2+x+1).

Here, "irreducible" just means "can't be factored further into linear factors." You can tell if a quadratic ax^2+bx+c is irreducible by using the quadratic formula to find the roots of $ax^2+bx+c=0$. There are real roots r_1 and r_2 is equivalent to saying the quadratic factors as $a(x-r_1)(x-r_2)$.

If we allowed imaginary roots $-$ *not of interest in this course* $-$ *then* ax^2+bx+c *would always have roots* r_1 *and* r_2 *(perhaps imaginary) and factor as* $a(x-r_1)(x-r_2)$

So, in theory, $Q(x) = (\ \ )(\ \ )(\ \ )\ldots(\ \ )$ where each factor ( ) is linear or irreducible quadratic. Some of these factors might be repeated.

$Q(x) = (x-3)(2x+1)$ or
$Q(x) = (x-3)(x-3)(2x+1) = (x-2)^2(2x+1)$ or
or
$Q(x) = (x-3)(2x+1)(x^2+x+1)^3(x^2+x+2)^2$

STEP 2) The appearance of the partial fraction decomposition depends on the "mix" of factors of $Q(x)$. We need to consider 4 different possibilities:

CASE I $\quad Q(x)$ has only linear factors and none of them is repeated
For example, $Q(x) = (x-3)(2x+1)$

CASE II $\quad Q(x)$ has only linear factors with one or more of them repeated
For example $Q(x) = (x-3)(x-3)(2x+1) = (x-3)^2(2x+1$

CASE III $\quad Q(x)$ has irreducible quadratic factors, and none of them is repeated.
For example, $Q(x) = (x-3)(2x+1)(x^2+x+1)(x^2+x+2)$

CASE IV $\quad Q(x)$ has irreducible quadratic factors with one or more of them repeated.
For example, $Q(x) = (x-3)(2x+1)(x^2+x+1)^3(x^2+x+2)^2$

CASE I In this case, the form of the partial fraction de composition is a sum of fractions like $\frac{\text{constant}}{\text{linear factor of } Q(x)}$, one fraction for each linear factor.

For example, if $\frac{P(x)}{Q(x)} = \frac{P(x)}{(x+2)\,(2x-7)(3x+5)\,(x-4)}$, then

$\frac{P(x)}{Q(x)} = \frac{A}{x+2} + \frac{B}{2x-7} + \frac{C}{3x+5} + \frac{D}{x-4}$ (and we need to determine $A, B, C,\ D$)

To return to the original example:

$\frac{2x^3-3x^2-8x-2}{x^2-2x-3} = 2x+1+\frac{1}{x^2-2x-3} \quad dx = \int 2x+1$

Focusing on the fraction: $\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

To find A and B, multiply by the least common denominator $(x-3)(x+1)$ to clear the fractions: we then get

$$1 = A(x+1) + B(x-3)$$

Method 1 to find A, B (combine terms)

$$1 = \ (A+B)x + (A-3B) = 1$$

so we need

$$\begin{cases} A+B & = 0 \\ A-3B & = 1 \end{cases}$$

Solve these equations to get $B = -\frac{1}{4}$, $A = \frac{1}{4}$

Method 2 (substitute convenient x values)

$$1 = A(x+1) + B(x-3)$$

Let $x = 3$ to get $1 = 4A$, so $A = \frac{1}{4}$
Let $x = -1$ to get $1 = -4B$, so $B = -\frac{1}{4}$

Either way, $\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{\frac{1}{4}}{x-3} + \frac{-\frac{1}{4}}{x+1}$ and each of these fractions is easy to integrate.

Finishing the original example: :

$$\int \frac{2x^3-3x^2-8x-2}{x^2-2x-3}\,dx \underset{\substack{\uparrow \\ \textit{long division}}}{=} \int 2x+1\;dx\,dx + \int \frac{1}{x^2-2x-3}\,dx$$

$$= x^2 + x + \int \frac{1}{x^2-2x-3}\,dx = x^2 + x + \int \frac{\frac{1}{4}}{x-3} + \frac{-\frac{1}{4}}{x+1}\;dx$$

$$= x^2 + x + \tfrac{1}{4}\ln|x-3| - \tfrac{1}{4}\ln|x+1| + C = x^2 + x + \tfrac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C$$

In case I, each fraction in the partial fraction decomposition can be integrated using a logarithm.

Note: $\int \frac{1}{x^2-2x-3}\,dx$ *could have been done without partial fractions: complete the square in the denominator and make an appropriate trig substitution.*

CASE II $Q(x)$ has only linear factors but one or more of them is repeated.
Then the form of the partial fraction decomposition is similar to case one except if a linear factor $(\alpha x + \beta)$ is repeated k times, it contributes k fractions to the decomposition with denominators $(\alpha x+\beta)$, $(\alpha x+\beta)^2$, ..., $(\alpha x+\beta)^k$.

For example, if $Q(x) = (2x+1)^2 \cdot$ (other linear factors), then the twice-repeated factor $(2x+1)^2$ contributes $\frac{\text{constant}}{(2x+1)} + \frac{\text{constant}}{(2x+1)^2}$ to the partial fraction decomposition. Here's a specific example of the form:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\underset{\uparrow}{x^3}(2x+1)^2(x-7)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(2x+1)} + \frac{E}{(2x+1)^2} + \frac{F}{x-7}$$

x^3 is the linear factor $(x-0)$ repeated 3 times

Example $\int \frac{x}{(x+1)^2(x-2)}dx = \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}\,dx$

Multiplying both sides by the least common denominator

gives $$x = A(x+1)(x-2) + B(x-2) + C(x+1)^2 \qquad (*)$$

Method 1: multiply out (*) on the right, collect terms, and set up a system of equations to solve for the unknowns A, B, C. (*) becomes

$$x = (A+C)x^2 + (-A+B+2C)x + (C-2A-2B)$$

so we need $\begin{cases} A \qquad\quad + C \quad = 0 \\ -A + B + 2C \quad = 1 \\ -2A - 2B + C \quad = 0 \end{cases}$

Solving gives $A = -\frac{2}{9}$, $B = \frac{1}{3}$, $C = \frac{2}{9}$

Method 2 (as in CASE I) Substitute convenient x values in (*). This is certainly easier than Method 1 in this example because we can clearly see some x values that make certain terms 0. It's then easy to find the unknowns:

Let $x = -1$ to get $-1 = A(0)(-2) + B(-3) + C(0)^2$ so $B = \frac{1}{3}$
Let $x = 2$ to get $2 = A(0) + B(0) + 9C$ so $C = \frac{2}{9}$

Now substitute these values for B, C and pick another x value (random, but simple), say $x = 0$:

In (*), let $x = 0$ to get $0 = -2A + \frac{1}{3}(-2) + \frac{2}{9}(1)^2$ so $A = -\frac{2}{9}$

Then $\int \frac{x}{(x+1)^2(x-2)}dx$
$$= \int \frac{-\frac{2}{9}}{x+1} + \frac{\frac{1}{3}}{(x+1)^2} + \frac{\frac{2}{9}}{x-2}\,dx$$
$$= -\frac{2}{9}\ln|x+1| - \frac{1}{3}\frac{1}{x+1} + \frac{2}{9}\ln|x-2| + C$$
$$= -\ln|x+1|^{2/9} - \frac{1}{3}\frac{1}{x+1} + \ln|x-2|^{2/9} + C$$
$$= \ln\left|\left(\frac{x-2}{x+1}\right)^{2/9}\right| - \frac{1}{3(x+1)} + C$$

Notice that in CASE II: the fractions in the decomposition that have a denominator with exponent 1 will integrate as logarithms; but when the exponent is bigger than 1,

$(k > 1)$ $\int \frac{\text{constant}}{(\alpha x+\beta)^k} dx = \int (\text{constant})(\alpha x + \beta)^{-k} dx$ which integrates as a power function.

Additional example, not done in class

$$\int \frac{1}{x(x-1)^2(x+3)} dx = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x+3)}\, dx$$

To determine A, B, C, D :

$$\frac{1}{x(x-1)^2(x+3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x+3)}$$

$$= \frac{A(x-1)^2(x+3) + Bx(x-1)(x+3) + Cx(x+3) + Dx(x-1)^2}{x(x-1)^2(x+3)}$$

We could (Method I) multiply out the numerator and set up a system of equations to solve for the unknown A, B, C, D. Method 2 (from CASE I) is certainly easier since we can obviously see some x's values that make certain terms 0. It's then easy to find 3 of the unknowns:

$$1 = A(x-1)^2(x+3) + Bx(x-1)(x+3) + Cx(x+3) + Dx(x-1)^2$$

Let $x = 1$	and get	$1 = 4C$	so $C = \frac{1}{4}$
Let $x = -3$	and get	$1 = -48D$	so $D = -\frac{1}{48}$
Let $x = 0$	and get	$1 = 3A$	so $A = \frac{1}{3}$

Substituting these values and some other (random, but simple) value, say $x = 2$, gives

$1 = \frac{1}{3}(1)(5) + 10B + (\frac{1}{4})(2)(5) - \frac{1}{48}(2)$, which gives
$1 = \frac{5}{3} + \frac{10}{4} - \frac{1}{24} + 10B$

$\frac{24 - 40 - 60 + 1}{24} = -\frac{75}{24} = 10B$, so $B = -\frac{75}{240} = -\frac{5}{16}$

So $\int \frac{1}{x(x-1)^2(x+3)} dx = \int \frac{\frac{1}{3}}{x} + \frac{-\frac{5}{16}}{x-1} + \frac{\frac{1}{4}}{(x-1)^2} + \frac{-\frac{1}{48}}{(x+3)}\, dx$

Notice that, as always in CASE II, each fraction will either integrate as a logarithm or a power function (*when the exponent in the denominator is bigger than* 1)

$$= \frac{1}{3}\ln|x| - \frac{5}{16}\ln|x-1) - \frac{1}{4}\frac{1}{x-1} - \frac{1}{48}\ln|x+3| + C$$

$$= \ln|x|^{1/3} - \ln|x-1|^{5/16} - \frac{1}{4}\frac{1}{x-1} - \ln|x+3|^{1/48} + C$$

$$= \ln\left|\frac{x^{1/3}}{(x-1)^{5/16}(x+3)^{1/48}}\right| - \frac{1}{4(x-1)} + C$$