Review

Q1. Using an integral, find the length of $y = 5x - 7$ for $10 \leq x \leq 12$.

A) $4\sqrt{3}$ **B) $\mathbf{2\sqrt{26}}$** C) $2\sqrt{6}$ D) 10 E) $2\sqrt{13}$

<u>Answer</u> $\frac{dy}{dx} = 5$, so length $= \int_{10}^{12} \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_{10}^{12} \sqrt{1 + 5^2}\, dx = 2\sqrt{26}$

If $y = f(t)$, and point $P = (a, f(a)$ is a point on the graph. Then

$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt =$ the length of the curve measured between the chosen starting point P and the point $(x, f(x))$. s is called the <u>arc length function for f measured from $(a, f(a))$.</u>

Q2. Write the integral that gives the length $s(x)$ (it depends on x!) of $y = 1 + t^{\frac{3}{2}}$ for $1 \leq t \leq x$.

A) $\int_1^x \sqrt{1 + \frac{3}{2} t^{\frac{1}{2}}}\, dt$ B) $\int_1^x \sqrt{\frac{9}{4} t}\, dt$ **C) $\boldsymbol{\int_1^x \sqrt{1 + \frac{9}{4} t}\, dt}$**

D) $\int_1^x \sqrt{\frac{3}{2} + \frac{9}{4} t}\, dt$ E) $\int_1^x \sqrt{\frac{9}{4} + \frac{9}{4} t^2}\, dt$

<u>Answer</u> $\frac{dy}{dt} = \frac{3}{2} t^{\frac{1}{2}}$, so $s(x) = \int_1^x \sqrt{1 + (\frac{dy}{dt})^2}\, dt = \int_1^x \sqrt{1 + \frac{9}{4} t}\, dt$

$$(= \ldots = \frac{1}{27}((4 + 9x)^{3/2} - 13^{3/2})$$

In this case, for example, $s(2) = \frac{1}{27}(22\sqrt{22} - 13\sqrt{13})$ gives the distance along the graph from the starting point $(1, 2)$ to the pont $(2,\ 1 + 2^{3/2}) = (2, 1 + 2\sqrt{2})$

In general, for $y = f(x)$, an arc length function looks like $s = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$ (where a determines the starting point $(a, f(a))$ on the graph from which distance along the curve is measured.)

Since s is a function of x, the Fundamental Theorem of Calculus tells us (*assuming f' is continuous*) that

$\frac{ds}{dx} = \sqrt{1 + (f'(x))^2}$ Since $\frac{ds}{dx} \geq 0$, the means that s is an <u>increasing</u> function of x : bigger x means bigger length $s(x)$ (*which should be instutitvely obvious*)

In differential notation, we can write this $ds = \sqrt{1 + (f'(x))^2}\, dx$

For a given value of x, this means:

if x is changed by a small amount $dx = \Delta x$, then s changed by an (exact) amount Δs, and $\Delta s \approx ds$ if dx is small.

Notice, that, not only is $\sqrt{1 + (f'(x))^2} \geq 0$ but actually $\sqrt{1 + (f'(x))^2} \geq 1$. *Therefore $ds \geq dx$: so when x changes by a small amount dx, s changes approximately by ds, and this $ds \geq dx$. (This should also be intuitively obvious: draw a picture and pick an x. If x changes by, say, $dx = 0.1$, you should see that the corresponding change in arc length is ≥ 0.1 – how <u>much</u> larger depends on the value of $\sqrt{1 + (f'(x))^2}$ and that in turn depends on "how steep" your graph is near x (as measured by the value of $f'(x)$); ds will <u>equal</u> dx only when $f'(x) = 0$ at your point x.)*

This is the same general interpretation of the symbols as we had in Math 131 when doing linear approximations with differentials:

if we had $y = f(x)$ and changed x by a small amount $dx = \Delta x$: then y changes by an (exact) amount Δy, and $\Delta y \approx dy$ if dx is small.

The lecture then imoved on to the topic of "apprpximate integration." We already know how to approximate an integral $\int_a^b f(x)\, dx$ using L_n, R_n, or M_n (the left endpoint, right endpoint, or midpoint rules). We introduced Simpson's Rule S_n which usually is more accurate than the other rules (for the same value of n).

We discussed in detail Simpson's Rule starting S_2 $(n = 2)$

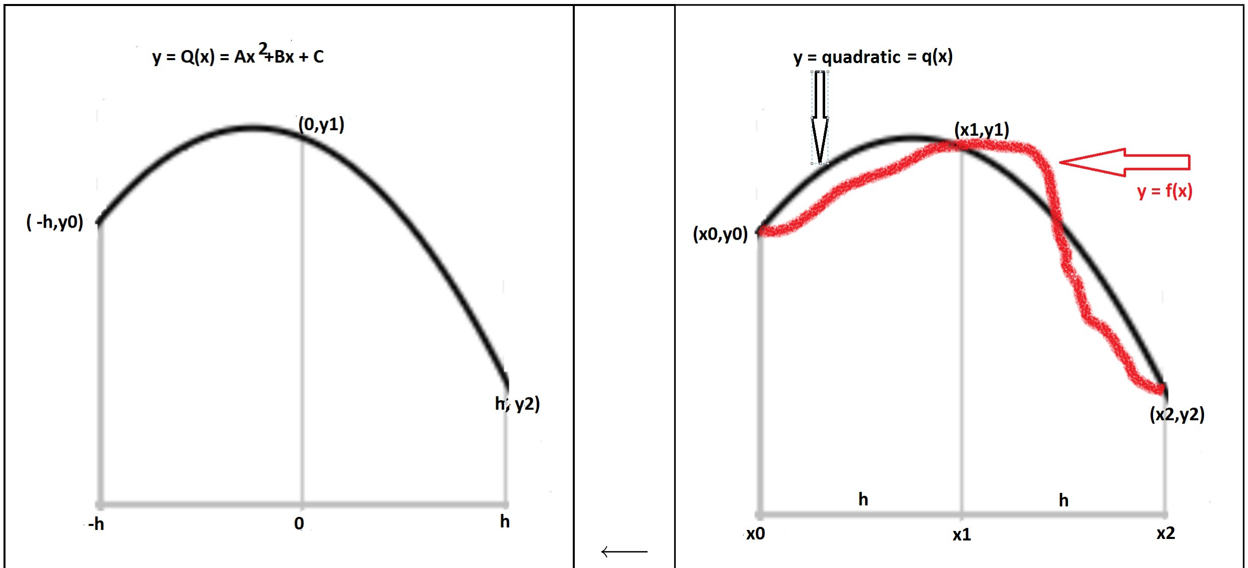
The interval $[a, b]$ is divided into 2 equal parts of length $\boldsymbol{\frac{b-a}{2} = \Delta x = h}$.
The endpoints of the subintervals are $a = x_0$, x_1, $x_2 = b$. The corresponding points on the graph of $y = f(x)$ are (x_0, y_0), (x_1, y_1), (x_2, y_2) where $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$.

We consider a quadratic function ("parabola") $y = q(x)$, through there three points and make an approximation:

$$\int_a^b f(x)dx \approx \int_a^b q(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$\uparrow$

($\textit{see picture and calculation below}$)



$\longleftarrow$ Translate left/right so x1 moves to 0

Starting on the right: the approximation is $\int_a^b f(x)dx = \int_a^b q(x)dx = ???$ *Translate the figure left/right (focusing only on the quadratic curve) so the* x_1 *moves to the origin. The* ***signed area*** *between the parabola and* x*-axis doesn't change* ***and*** y_0, y_1, y_2 ***do not change*** *between the pictures! So (on the left):*

$$\int_{-h}^{h} Ax^2 + Bx + C\,dx \underset{\substack{\uparrow \\ (\textit{see below})}}{=} \frac{h}{3}(y_0 + 4y_1 + y_2) = \int_{x0}^{x_2} q(x)\,dx$$

$$\int_{-h}^{h} Ax^2 + Bx + C\,dx = \underbrace{\int_{-h}^{h} {\color{red}Ax^2 + C}\,dx}_{\color{red}\text{even function}} + \underbrace{\int_{-h}^{h} {\color{blue}Bx}\,dx}_{\color{blue}\text{odd function}} =$$

$$= 2\int_0^h {\color{red}Ax^2 + C}dx + {\color{blue}0}$$

$$= 2(A\tfrac{x^3}{3} + Cx)\Big|_0^h = 2(A\tfrac{h^3}{3} + Ch) = \tfrac{h}{3}(2Ah^2 + 6C)$$

We know that
$$\begin{cases} y_0 = Q(-h) = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C \\ y_1 = Q(0) \qquad\qquad\qquad\qquad\qquad\quad = \qquad\qquad\;\; C \\ y_2 = Q(h) \qquad\qquad\qquad\qquad\qquad\quad = Ah^2 + Bh + C \end{cases}$$

Add Equation 1 + 4(Equation 2) + Equation 3: $2Ah^2 + 6C = y_0 + 4y_1 + y_2$

so

$$\int_{-h}^{h} Ax^2 + Bx + C\,dx = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

For Simpson's Rule S_n in general (see textbook, and worked out in class):

n must be <u>even</u>: the interval $a, b]$ is divided into n equal parts of length $h = \Delta x = \frac{b-a}{n}$

$$a = x_0 < x_1 < x_2 < .... < \; < x_{n-2} < x_{n-1} < x_n = b$$

Since n is even, we can “group" the subinteverals as

$$[x_0, x_2],\ [x_2, x_4],\ [x_4, x_6],\ \ldots,\ [x_{n-2}, x_n]$$

and do a parabolic approximation to f on each each of these subintervals. (The case of S_2 one subinterval $[x_0, x_2]$ $-$ was shown above)

The result, from class: $\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)dx\ + \int_{x_2}^{x_4} f(x)\,dx + ... + \int_{x_{n-2}}^{x_n} f(x)\,dx$

$$\int_a^b f(x)\,dx \qquad = \int_{x_0}^{x_2} f(x)dx\ + \int_{x_2}^{x_4} f(x)\,dx + ... + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$\approx \int_{x_0}^{x_2}(\text{quadratic})\,dx\ + \int_{x_2}^{x_4}(\text{quadratic})\,dx + ... + \int_{x_{n-2}}^{x_n}(\text{quadratic})\,dx$$

$$= \tfrac{h}{3}(y_0 + 4y_1 + y_2) + \tfrac{h}{3}(y_2 + 4y_3 + y_4) + ... + \tfrac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \tfrac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ...\ +\ 4y_{n-1} + y_n) = S_n$$

<u>Example</u> Use Simpson's Rule S_4 to approximate the value of $\int_2^3 \frac{1}{x+1}dx$. (*This integral is simple enough that we don't need to make an approximate $-$ the <u>exact value</u> is* $\ln(x+1)\Big|_2^3 = \ln 4 - \ln 3 = \ln\frac{4}{3}$ (≈ 0.287682 , rounded to 6 decimal places). But knowing the exact value, we can compare it to what aproximation is made by S_4.

$h = \frac{1}{4} = \Delta x$; endpoints of the subintervals are

$$2 < \tfrac{9}{4} < \tfrac{5}{2} < \tfrac{11}{4} < 3$$

$$\int_2^3 \frac{1}{x+1}dx \approx S_4 = \frac{\frac{1}{4}}{3}(f(2) + 4f(\tfrac{9}{4}) - 2f(\tfrac{5}{2}) + 4f(\tfrac{11}{4}) + f(3)) \approx \quad 0.287682.$$

(rounded to 6 decimal places).

Rounded to 6 decimal places, the “error” in approximating the integral by S_4 is only 0.000001!