Suppose we want to approximate $\int_a^b f(x)\,dx$

We subdivide $[a, b]$ into n equal subintervals of width Δx. The subintervals are

|_____|_____|_ ··· _|_____|_ ··· _____|_____|

x_0 x_1 x_2 x_{i-1} x_i x_{n-1} x_n

|| ||

a b

The midpoints of the subintervals are denoted by $\overline{x_i}$

Q1: For the Midpoint Rule, the approximation is

$M_n = \;?\; \cdot\; (\blacksquare f(\overline{x_1}) + \blacksquare f(\overline{x_2}) + \blacksquare f(\overline{x_3}) ... + ... + \blacksquare f(\overline{x_n}))$ where

A) $? = \Delta x,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 2, 1,\ ... , 2, 1$

B) $? = \Delta x/2,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 2, 1,\ ... , 2, 1$

C) $? = \Delta x,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 1, 1,\ ... , 1, 1$

D) $? = \Delta x/3,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 4, 2, 4,\ ... , 2, 1$

E) $? = \Delta x/3,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 4, 2, 4, 2, ... , 4, 1$

Answer C, boldfaced

Q2: For Simpson's Rule (only for even n), the approximation is

$S_n = \;?\; \cdot\; (\blacksquare f(x_0) + \blacksquare f(x_1) + \blacksquare f(x_2) + ... + \blacksquare f(x_n))$ where

A) $? = \Delta x,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 2, 1,\ ... , 2, 1$

B) $? = \Delta x/2,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 2, 1,\ ... , 2, 1$

C) $? = \Delta x,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 1, 1,\ ... , 1, 1$

D) $? = \Delta x/3,$ and the pattern of coefficients $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 4, 2, 4, 2,\ ... , 2, 1$

E) $? = \Delta x/3,$ and the coefficient pattern $\blacksquare,\blacksquare,\blacksquare......,\blacksquare$ is $1, 4, 2, 4, 2\ , ... , 4, 1$

Answer E), boldfaced

Approximating $\int_a^b f(x)\,dx$

Divide $[a,b]$ into n subintervals, each of width $\Delta x = \frac{b-a}{n}$
The n subintervals are

$$\begin{array}{ccccccc} a & & & & & & b \\ || & & & & & & || \\ [x_0, x_1], & [x_1, x_2], & \ldots, & [x_{i-1}, x_i], & \ldots, & & [x_{n-1}, x_n] \\ \uparrow & \uparrow & & \uparrow & & \uparrow & \\ \overline{x_1} & \overline{x_2} & & \overline{x_i} & & \overline{x_n} & \end{array}$$

Midpoints are $\overline{x_1}$ $\overline{x_2}$ $\overline{x_i}$ $\overline{x_n}$

<u>Midpoint Rule</u>:

$$M_n = \Delta x(f(\overline{x_1}) + \ldots + f(\overline{x_i}) + \ldots + f(\overline{x_n}))$$

<u>Simpson's Rule</u> (n must be <u>even</u>) :

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n))$$

Example: $\int_0^1 \cos x\,dx$ (an easy integral to evaluate <u>exactly</u>, but chosen so we can compare the exact value to the approximate values : $\int_0^1 \cos x\,dx = \sin 1$)

With $n = 4$: Subintervals are $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$
$\Delta x = \frac{1}{4}$

The <u>midpoints</u> are $\overline{x_1} = \frac{1}{8}$, $\overline{x_2} = \frac{3}{8}$, $\overline{x_3} = \frac{5}{8}$, $\overline{x_4} = \frac{7}{8}$

$$M_4 = \frac{1}{4}(f(\tfrac{1}{8}) + f(\tfrac{3}{8}) + f(\tfrac{5}{8}) + f(\tfrac{7}{8}) = 0.843666$$

(<u>rounded</u> to 6 decimal places)

$$S_4 = \frac{1/4}{3}(f(0) + 4f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 4f(\tfrac{3}{4}) + f(1))$$

$$= \frac{1}{12}(\cos(0) + 4\cos(\tfrac{1}{4}) + 2\cos(\tfrac{1}{2}) + 4\cos(\tfrac{3}{4}) + \cos(1)) = 0.841489$$

(<u>rounded</u> to 6 decimal places)

Error Estimates:

In general, for $\int_a^b f(x)dx$

error

$$\int_a^b f(x)dx = M_n + E_M$$

where K is chosen so $|f''(x)| \le K$ on $[a, b]$

$$|\int_a^b f(x)dx - M_n| = |E_M| \le \frac{K(b-a)^3}{24n^2}$$

<u>and</u>

error

$$\int_a^b f(x)dx = S_n + E_S$$

where K is chosen so $|f^{(\text{iv})}(x)| \le K$ on $[a, b]$

$$|\int_a^b f(x)dx - S_n| = |E_S| \le \frac{K(b-a)^5}{180n^4}$$

These "error bound formulas" give us a handle on error size:

"the magnitude of the error in the approximation" $\le$ ....

$\uparrow$

is **<u>at most</u>** ...

<u>Examples:</u>

1) $\int_0^1 \cos x\, dx = M_4 + E_M$

can choose $K = 1$ since
$|f''(x)| = |-\cos x| \le 1$ on $[0, 1]$

$\downarrow$

$$|\int_0^1 \cos x\, dx - M_4| = |E_M| \le \frac{K(1-0)^3}{24n^2} = \frac{1}{24n^2} = \frac{1}{24 \cdot 4^2} = 0.002604$$

(rounded to 6 decimal places)

so $\quad -0.002604 \le \int_0^1 \cos x\, dx - M_4 \le 0.002604$

$$0.843666 - 0.002604 \le \int_0^1 \cos x\, dx \le 0.843666 + 0.002604$$

so $\quad 0.841062 \le \int_0^1 \cos x\, dx \le 0.846270$

$\uparrow$

exact value $=$ sin 1
($= 0.841471$, rounded to 6 decimal places)

2) $\int_0^1 \cos x \, dx = S_4 + E_S$

can choose $K = 1$ since $|f^{(\text{iv})}(x)|$
$|f^{(\text{iv})}(x)| = |-\cos x| \leq 1$ on $[0, 1]$
$\downarrow$

$|\int_0^1 \cos x \, dx - M_4| = |E_M| \leq \frac{K(1-0)^3}{180n^4} = \frac{1}{180 \cdot 4^4} = 0.000022$
(rounded to 6 decimal places)

so $-0.000022 \leq \int_0^1 \cos x \, dx - S_4 \leq 0.000022$

$S_4 - 0.000022 \leq \int_0^1 \cos x \, dx \leq S_4 + 0.000022$

$0.841489 - 0.000022 \leq \int_0^1 \cos x \, dx \leq 0.841489 + 0.000022$

$0.841467 \leq \int_0^1 \cos x \, dx \leq 0.841511$
$\uparrow$
exact value $= \sin 1$
$(= 0.841471,$ rounded to 6 decimal places)

How large must n be to guarantee that $\left|\int_0^1 \cos x\,dx - M_n\right| < 10^{-6}$?

This will be true if $\left|\int_0^1 \cos x\,dx - M_n\right| \leq \frac{1(1-0)^3}{24n^2} < 10^{-6}$

$$24n^2 > 10^6$$

so we need $\quad n > \sqrt{\frac{10^6}{24}} \approx 204.1$

Since n is an integer, $\quad n = 205$ is guaranteed to work.

How large must n be to guarantee that $\left|\int_0^1 \cos x\,dx - S_n\right| < 10^{-6}$?

This will be true if $\left|\int_0^1 \cos x\,dx - S_n\right| \leq \frac{1(1-0)^5}{180n^4} < 10^{-6}$

$$180n^4 > 10^6$$

so we need $\quad n > \sqrt[4]{\frac{10^6}{180}} \approx 8.6$

Since n must be
an <u>even</u> integer $\quad n = 10$ is guaranteed to work.