Error bound $E_S$ for Simpson's Rule ($n$ even)

$|\int_a^b f(x)\,dx - S_n) \le \frac{K(b-a)^5}{180n^4}$ where $K$ must be chosen so that $|f^{(iv)}(x)| \le K$ on $[a,b]$

Q1: If $|f^{(iv)}(x)| \le 4$ on $[0,4]$, then $|\int_0^4 f(x)dx - S_4| \le$ ?

**A)** $\mathbf{\frac{4}{45} \approx 0.089}$ B) $\frac{2}{45} \approx 0.044$ C) $\frac{7}{180} \approx 0.039$ D) $\frac{24}{180} \approx 0.133$

E) $\frac{28}{180} \approx 0.156$

Answer: $|\int_0^4 f(x)dx - S_4| \le \frac{K(b-a)^5}{180n^4}$. We are told that we can use $K = 4$.

So $|\int_0^4 f(x)dx - S_4| \le \frac{4(4-0)^5}{180(4)^4} = \frac{4^6}{180(4^4)} = \frac{4^2}{180} = \frac{4}{45}$

Q2: We use Simpson's Rule with $n = 2$ to approximate $\int_{-3}^{17} x^3 - 4x^2 + 13x - 7\,dx$.

We can then say that $|\int_{-3}^{17} x^3 - 4x^2 + 13x - 7\,dx \;-\; S_2| \le$ ?

A) $\frac{1}{20}$ B) $\frac{1}{10}$ C) $\frac{1}{5}$ D) $\frac{1}{2}$ E) 0

Answer For $f(x) = x^3 - 4x^2 + 13x - 7$ (or any cubic polynomial)

$f'(x) =$ quadratic polynomial (degree 2)
$f''(x) =$ linear polynomial (degree 1)
$f'''(x) =$ constant polynomial ("degree 0")
and $f^{(iv)}(x) = 0$

So for this example (or $\int_a^b$(any cubic polynomial) $dx$) we can choose $K = 0$, and get

$|\int_a^b$(cubic polynomial) $dx - S_2| \le \frac{0\,(b-a)^5}{180(2)^2} = 0$

This says the magnitude of the error for Simpson's rule here, even with $n = 2$, is $0$: so the error is 0 – meaning that $\int_a^b$(cubic polynomial) $dx = S_2$ (exactly!)

To illustrate with a numerically simpler example: $\int_0^2 x^3 dx = \frac{x^4}{4}\Big|_0^2 = \frac{16}{4} = 4$ (exact value!)
and Simpson $S_2 = \frac{1}{3}(0^3 + 4(1)^3 + 2^3) = \frac{12}{3} = 4$! $S_2$ "nails it" exactly!

Example Estimate $\int_1^2 f(x)\,dx = \int_1^2 e^{\sin x}\,dx$ and error using $S_6$.

Calculate $S_6$ : $\Delta x = \frac{1}{6}$

$$S_6 = \frac{1/6}{3}\left(f(1) + 4f(\tfrac{7}{6}) + 2f(\tfrac{8}{6}) + 4f(\tfrac{9}{6}) + 2f(\tfrac{10}{6}) + 4f(\tfrac{11}{6}) + f(\tfrac{12}{6})\right)$$

$$= 2.6046953633 \ (\textit{rounded})$$

Estimate error: Recall that for any numbers $a, b$

$$|a+b| \le |a| + |b|$$

This also gives us

$$|a-b| = |a + (-b)| \le |a| + |(-b)| = |a| + |b|$$

Some calculation (perhaps assisted by something like Wolfram Alpha) gives

$$f^{(\text{iv})}(x) = \frac{e^{\sin x}}{8}(-4\sin x - 12\sin 3x - 24\cos 2x + \cos 4x - 1)$$

so taking $|\quad|$ and using the inequalities above gives

$$|f^{(\text{iv})}(x)| \le \frac{e^{\sin x}}{8}(4|\sin x| + 12|\sin 3x| + 24|\cos 2x| + |\cos 4x| + 1)$$

Since $|\sin x| \le 1$ always, its guaranteed that $\frac{e^{\sin x}}{8} \le \frac{e^1}{8} = \frac{e}{8}$ , so

$$\le \frac{e}{8}(4 + 12 + 24 + 1 + 1| = \frac{42e}{8} = \frac{21e}{4}$$

Therefore we can use $K = \frac{21e}{4}$ to get

$$|\int_1^2 e^{\sin x}\,dx - S_6| \le \frac{K(b-a)^5}{180n^4} = \frac{\frac{21e}{4}(2-1)^5}{180(6^4)} = .0000611753 \ (\textit{rounded}), \text{ or}$$

so $$-.0000611753 \le \int_1^2 e^{\sin x}\,dx - S_6 \le .0000611753$$

so

$$S_6 - .0000611753 \le \int_1^2 e^{\sin x} dx \le S_6 + .0000611753$$

which works out to

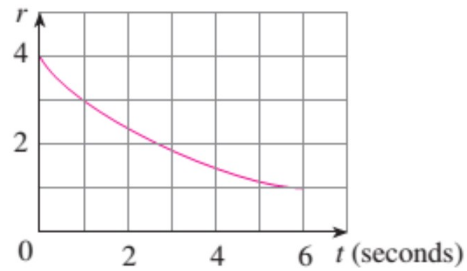
$$2.6046953633 - .0000611753 \le \int_1^2 e^{\sin x} dx \le 2.6046953633 + .0000611753$$

$$2.6046341880 \le \int_1^2 e^{\sin x} dx \le 2.6047565386$$

(For comparison: to 5 decimal places, Wolfram Alpha (*using who-knows-what method, but likely very accurate* ) gives the value $\int_1^2 e^{\sin x} dx = 2.60466$ (*rounded to 5 decimal places by default*)

Example where approximating the integral is necessary because we don't have a formula for the function we want to integrate, only graphical or numerical information.

Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of $r$ is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first 6 hours.




The total amount leaked (New Change Theorem!) is $\int_0^6 r(t)\,dt$ which we can approximate from the graph using, say $n = 6$ with $\Delta x$=1:

Visually estimate

| $t$ | $r(t)$ |
|---|---|
| | |
| 0 | 4 |
| 1 | 3 |
| 2 | $\frac{7}{3}$ |
| 3 | $\frac{15}{8}$ |
| 4 | $\frac{3}{2}$ |
| 5 | $\frac{9}{8}$ |
| 6 | 1 |

Amount of water in tank at time $t$ : $A(t)$

$$\frac{dA}{dt} = r(t)$$

Total amount leaked

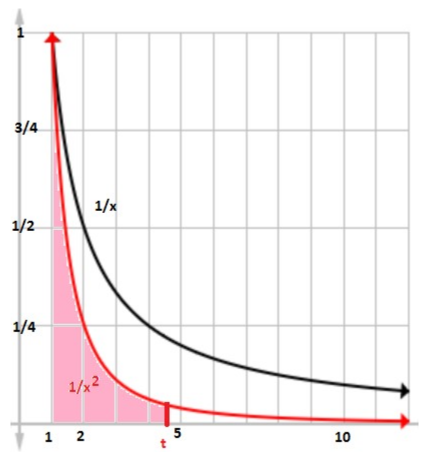
$$= \int_0^6 r(t)dt$$

$$\approx\ S_6 = \tfrac{1}{3}(r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)) \approx 12.2\ \text{L}$$

The lecture ended with a brief introduction to one kind of improper integral

Consider $\int_1^\infty \frac{1}{x^2}dx$ and $\int_1^\infty \frac{1}{x}\,dx$

These are example of one kind of "improper integral": called "improper" because we are integrating over an "infinitely long" interval $[1, \infty)$ rather than over some interval of finite length (such as $[2, 7]$ for the integral $\int_2^7 \frac{1}{x^2}dx$)



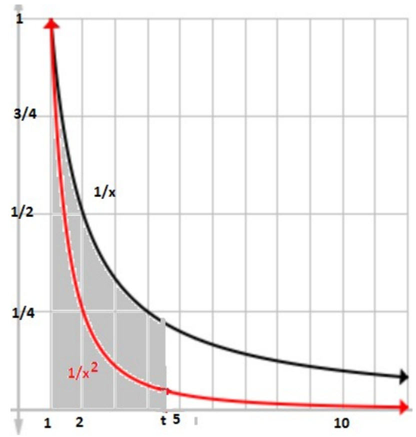

We don't know how to do $\int_1^\infty \frac{1}{x^2}dx$. The definition is that we do the thing that we do know how to do first compute $\int_1^t \frac{1}{x^2}dx$ (representing the shaded area), then let $t \to \infty$.

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t = \lim_{t\to\infty} \underbrace{\mathbf{1-\frac{1}{t}}}_{\text{the shaded area}} = 1$$

so we say that $\int_1^\infty \frac{1}{x^2}dx$ exists because that limit exists. We also sometimes express the same thing by saying that $\int_1^\infty \frac{1}{x^2}dx$ converges to value 1.

We can think of 1 as being (?paradoxically?) the area of the infinite region under the graph of $f(x) = \frac{1}{x^2}$ over the interval $[1, \infty)$

Now consider $\int_1^\infty \frac{1}{x}\, dxv$




Using the same idea: we first compute $\int_1^t \frac{1}{x}\, dx$ (representing the shaded area) and then let $t \to \infty$ :

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx = \lim_{t\to\infty} \ \ln|x|\,\Big|_1^t = \lim_{t\to\infty} \ \underbrace{\ln t}_{\text{the shaded area}}, \text{ and this limit d.n.e.}$$

Here, the limit doesn't exist (since $\ln t \to \infty$ as $t \to \infty$). We say that the improper integral $\int_1^\infty \frac{1}{x}\, dx$ doesn't exist. Sometimes we express the same thing by saving $\int_1^\infty \frac{1}{x}\, dx$ diverges.

An improper integral (*see preceding example*) is said to exist (converge) only when the limit in its definition is a number. As in Math 131, we might write $\lim_{t\to\infty} \ln t = \infty$ as a way of saying why the limit doesn't exist, and therefore we might write $\int_1^\infty \frac{1}{x}\, dx = \infty$ as a way of expressing why $\int_1^\infty \frac{1}{x} dx$ doesn't exist. $\int_1^\infty \frac{1}{x}\, dx = \infty$ is a way of saying that the integral diverges in a certain way.

Geometrically, this all means that the area under $f(x) = \frac{1}{x}$ over the interval $[1, \infty)$ doesn't exist (or we could say, area $= \infty$). This is in sharp contrast to our calculation of the area under $\frac{1}{x^2}$ over $[1.\infty)$.

*Roughly you need to think the height of the graph $\frac{1}{x^2}$ shrinks toward height $0$ fast enough that the area under the graph over $[1, \infty)$ can come out a finite real number. But that the height of $\frac{1}{x}$ is higher and doesn't shrink fast enough for the area under the graph over $[1, \infty)$ to come out a finite number.*

*Try to retrain your intuition here: you should be comfortable with the integral calculations from what we've already learned. then you try to "recalibrate" you intuition about areas to take these new observations into account.*