In the last lecture, we looked quickly at two examples of improper integrals (repeated below). They are both TYPE I improper integrals. These are integrals where the integral is computed over an “infinitely long' interval. The 3 ways this happens are shown in parts a),b), c) of the definition.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

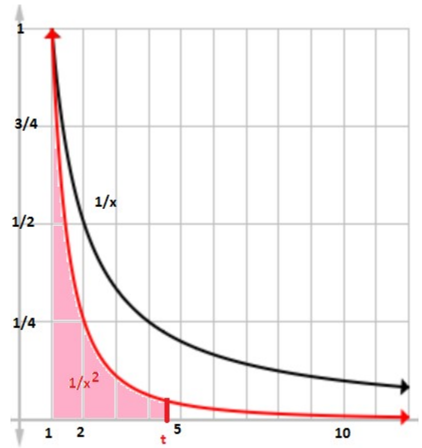
(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Examples (from preceding lecture):

1) $\int_1^\infty \frac{1}{x^2} dx$



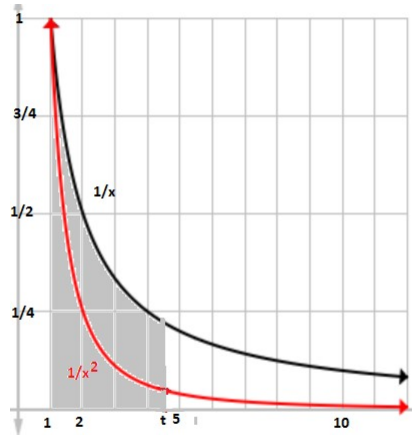
The definition is that we first do something that we know how to do: compute $\int_1^t \frac{1}{x^2} dx$ (which represents the shaded area), then let $t \to \infty$.

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_1^t = \lim_{t\to\infty} \underbrace{\left(1 - \frac{1}{t}\right)}_{\text{the shaded area}} = 1$$

We say that $\int_1^\infty \frac{1}{x^2} dx$ exists because that limit exists. We also sometimes express the same thing by saying that $\int_1^\infty \frac{1}{x^2} dx$ converges to value 1.

We can think of 1 as being (?paradoxically?) the area of the infinite region under the graph of $f(x) = \frac{1}{x^2}$ over the interval $[1, \infty)$

2) Now consider $\int_1^\infty \frac{1}{x}\,dx$



Using the same idea: we first compute $\int_1^t \frac{1}{x}\,dx$ (representing the shaded area) and then let $t \to \infty$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ \ln|x|\,\Big|_1^t \ = \ \lim_{t\to\infty} \ \underbrace{\ln t}_{\uparrow \text{ the shaded area}}$$, and this limit d.n.e.

Here, the limit doesn't exist (since $\ln t \to \infty$ as $t \to \infty$). We say that the improper integral $\int_1^\infty \frac{1}{x}\,dx$ doesn't exist. Sometimes we express the same thing by saving $\int_1^\infty \frac{1}{x}\,dx$ diverges.

An improper integral (*see preceding examples*) is said to exist (converge) only when the limit in its definition is a number. As in Math 131, we might write might write $\int_1^\infty \frac{1}{x}\,dx = \infty$ as a way of expressing why $\int_1^\infty \frac{1}{x}dx$ doesn't exist. $\int_1^\infty \frac{1}{x}\,dx = \infty$ is a way of saying that the integral diverges in a certain way.

Geometrically, this all means that the area under $f(x) = \frac{1}{x}$ over the interval $[1, \infty)$ doesn't exist (or we could say, area $= \infty$). This is in sharp contrast to our calculation of the area under $\frac{1}{x^2}$ over $[1.\infty)$.

Another seeming "paradox of the infinite" : suppose the (infinite) area under $f(x) = \frac{1}{x}$ $(0 \le x < \infty)$ is revolved around the x-axis. Using the method of disks/washers, the resulting solid has volume $V = \int_1^\infty \pi f^2(x)\,dx = \int_1^\infty \pi(\frac{1}{x})^2\,dx = \pi\int_1^\infty \frac{1}{x^2}dx$ $= \pi(1) = \pi$. This volume is finite even though the area being revolved is infinite !!

Q1: If $p < 1$, what is $\lim\limits_{t\to\infty} \frac{t^{1-p}}{1-p}$?

A) 0 B) 1 C) $\frac{1}{1-p}$ D) $\frac{1}{p-1}$ **E) DNE**

<u>Answer</u> If $p < 1$, the exponent $1-p > 0$, and $\lim\limits_{t\to\infty} t^{(\text{positive exponent})}$ DNE $(=\infty)$.

Q2: If $p > 1$, what is $\lim\limits_{t\to\infty} \frac{t^{1-p}}{1-p}$?

A) 0 B) 1 C) $\frac{1}{1-p}$ D) $\frac{1}{p-1}$ E) DNE

<u>Answer</u> If $p > 1$, the exponent $1-p < 0$, and $\lim\limits_{t\to\infty} t^{(\text{negative exponent})}$

$= \lim\limits_{t\to\infty} \frac{1}{t^{(\text{positive exponent})}} = 0.$

More Examples:

3) $\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \underline{\text{converges,}} & \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} & \text{if } p > 1 \\ \underline{\text{diverges,}} & \int_1^\infty \frac{1}{x^p}\,dx = \infty & \text{if } p \le 1 \end{cases}$

Why? The case where $p = 1$ (integral diverges) is Example 2.

If $p < 1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx = \lim\limits_{t\to\infty} \int_1^t \frac{1}{x^p}dx = \lim\limits_{t\to\infty} \int_1^t x^{-p}dx =$

exponent positive!

$$= \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \qquad \text{DNE } (=\infty)$$

If $p > 1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx = \lim\limits_{t\to\infty} \int_1^t \frac{1}{x^p}dx = \lim\limits_{t\to\infty} \int_1^t x^{-p}dx =$

exponent negative!

$$= \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= 0 - \frac{1}{1-p} = \frac{1}{p-1} \quad \text{DNE } (=\infty)$$

This agrees with Example 1) where $p = 2$:

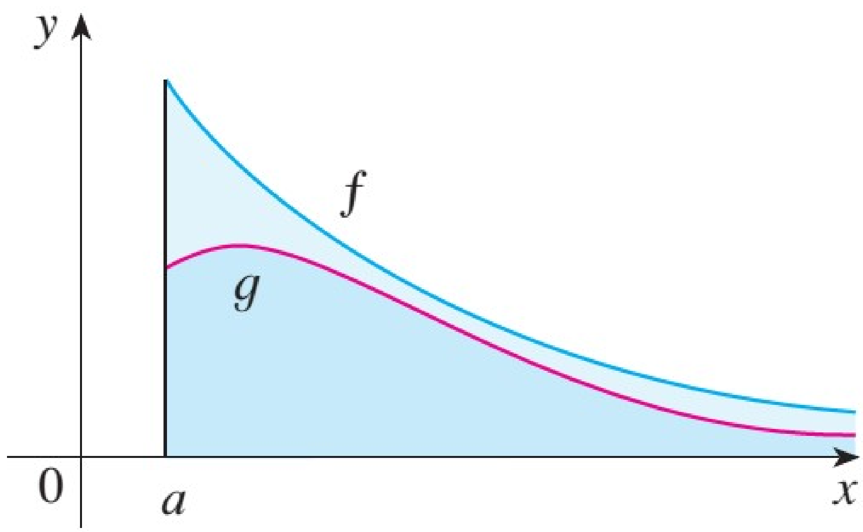
$$\int_1^\infty \frac{1}{x^2} = \frac{1}{2-1} = 1.$$

Sometimes we can decide wheter an integral converges or diverges (without actually computing it) just by comparing its size to the size of another more familiar integral whose behavior we know.

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Example 4 (some comparisons)

On the interval $[1, \infty)$

$$0 < x - \tfrac{1}{2} < x < x^2 < x^2 + 17 \qquad \text{so}$$

$$\frac{1}{x-\frac{1}{2}} > \frac{1}{x} > \frac{1}{x^2} > \frac{1}{x^2+17}$$

By comparison:

$0 < \frac{1}{x^2+17} < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ converges, so $\int_1^\infty \frac{1}{x^2+17}\,dx$ also converges

$\frac{1}{x} > \frac{1}{x-\frac{1}{2}} > 0$ and $\int_1^\infty \frac{1}{x} dx$ diverges, so $\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$ also diverges.

Notice that if the integral of the larger function, $\int_a^\infty f(x)\,dx$, diverges, no conclusion is possible about the integral of the smaller function $\int_1^\infty g(x)dx$:

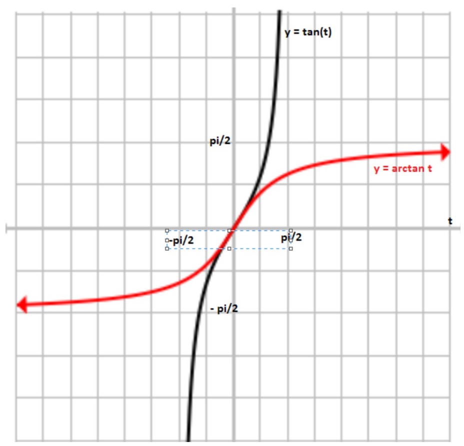
For example, on $[1,\infty)$ $\quad \sqrt{x} < x < x^2$ so $\frac{1}{\sqrt{x}} > \frac{1}{x} > \frac{1}{x^2}$

$\frac{1}{\sqrt{x}} > \frac{1}{x}$, $\int_1^\infty \frac{1}{\sqrt{x}}dx \ = \int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges and $\int_1^\infty \frac{1}{x}dx$ also diverges

$\frac{1}{\sqrt{x}} > \frac{1}{x^2}$, $\int_1^\infty \frac{1}{\sqrt{x}}dx \ = \int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges and $\int_1^\infty \frac{1}{x^2}dx$ converges

Similarly, if the integral of the smaller function $\int_1^\infty g(x)dx$ converges, no conclulsion is possible about the integral of the larger function $\int_1^\infty f(x)dx$.

5) Recall the graphs:



$\int_{-\infty}^{1} \frac{1}{1+x^2}dx$ is a Type I imroper integral of the kind b) in the definition above.

$$\int_{-\infty}^{1} \frac{1}{1+x^2}dx = \lim_{t\to-\infty} \int_t^1 \frac{1}{1+x2}\,dx = \lim_{t\to-\infty} \arctan x\Big|_t^1 = \lim_{t\to-\infty} (\arctan 1 - \arctan t)$$

$$= \tfrac{\pi}{4} - (-\tfrac{\pi}{2}) = \tfrac{3\pi}{4} \quad (\textit{integral converges, value } \tfrac{3\pi}{4})$$

$\uparrow$

see graph!

6) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$ is a Type I imroper integral of the kind c) in the definition above.

Break integral into two parts at some (*arbitrary*) point a, say $a = 1$: the choice of a will not affect your conclusion about whether the integral converges or diverges, nor, if the integral converges, your conclusion about the value of the integral.)

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{1} \frac{1}{1+x^2}dx + \int_1^\infty \frac{1}{1+x^2}dx.$ *Examine both parts separately.*

- $\int_1^\infty \frac{1}{1+x^2}\,dx \quad = \lim_{t\to\infty}\int_1^t \frac{1}{1+x2}\,dx = \lim_{t\to\infty} \arctan x\Big|_1^t = \lim_{t\to\infty}(\arctan t - \arctan 1)$

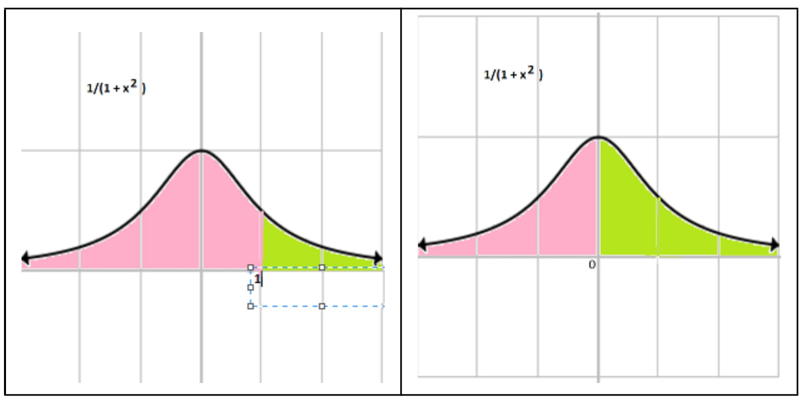
  $= \frac{\pi}{2} - (\frac{\pi}{4}) = \frac{\pi}{4} \quad$ (*integral converges, value* $\frac{\pi}{4}$)

and
- $\int_{-\infty}^1 \frac{1}{1+x^2}\,dx \; = \frac{3\pi}{4}$ (*see example 5*)

Since both integrals separately converge,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^1 \frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{1+x^2}\,dx = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$$

Notice that if someone had used 0 rather than 1 to "break" the original intebgrsal into two pieces, the final conclusion wouldn't change. In terms of area:



$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^1 \frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{1+x^2}\,dx \;=\; \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \pi$$

$$= \frac{3\pi}{4} \qquad + \qquad \frac{\pi}{4} \qquad = \qquad \frac{\pi}{2} \qquad + \qquad \frac{\pi}{2}$$