Review of definition of imp[roper integrals, Type I   (infinitely long intevral of integration_

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Q1 Calculate $\lim\limits_{t\to\infty}\int_{-t}^{t} x\,dx$

A) d.n.e. $(=\infty)$ B) d.n.e. $(=\ -\infty)$ C) -1

D) 0 E) 1

$\int_{-t}^{t} x\,dx = \frac{x^2}{2}\Big|_{-t}^{t} = 0$, so $\lim\limits_{t\to\infty}\int_{-t}^{t} x\,dx = \lim\limits_{t\to\infty} 0 = 0$

<u>NOTE</u>: by definition, above, $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

Since $\int_{0}^{\infty} x\,dx = \lim\limits_{t\to\infty}\frac{x^2}{2}\Big|_{0}^{t}$ d.n.e. $(=\infty)$, $\int_{0}^{\infty} x\,dx$ <u>diverges</u>. Therefore we says that $\int_{-\infty}^{\infty} x\,dx$ <u>diverges</u>.

$\int_{-\infty}^{\infty} x\,dx$ is <u>not</u> the same as $\lim\limits_{t\to\infty}\int_{-t}^{t} x\,dx$

Q2 For each integral write C ("convergent") or D ("divergent")

$\int_{1}^{\infty}\frac{1}{\sqrt{x}}\,dx$, $\int_{1}^{\infty}\frac{1}{x^{1.0000001}}\,dx$, $\int_{3}^{\infty}\frac{1}{x}\,dx$, $\int_{2}^{\infty}\frac{2x}{x^2+1}\,dx$, $\int_{1}^{\infty}\frac{1}{x^{0.9999999}}\,dx$

A) The integrals are: D, C, D, C, D

B) The integrals are: D, C, D, D, D

C) The integrals are: D, D, D, C, C

D) The integrals are: C, D, C, C, D

E) The integrals are: D, C, D, C, C

From the preceding lecture, $\int_{1}^{\infty}\frac{1}{x^p}\,dx \begin{cases} \underline{\text{converges}}, & \int_{1}^{\infty}\frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if } p>1 \\ \underline{\text{diverges}}, & \int_{1}^{\infty}\frac{1}{x^p}\,dx = \infty \quad \text{if } p\le 1 \end{cases}$

Therefore $\int_{1}^{\infty}\frac{1}{\sqrt{x}}\,dx$ diverges, $\int_{1}^{\infty}\frac{1}{x^{1.0000001}}\,dx$ converges, $\int_{3}^{\infty}\frac{1}{x}\,dx$ diverges, $\int_{1}^{\infty}\frac{1}{x^{0.9999999}}\,dx$ diverges.

Notice how the tny change in the function between $\int_1^\infty \frac{1}{x^{0.9999999}}\,dx$ and $\int_1^\infty \frac{1}{x^{1.0000001}}\,dx$ changes the behavior of the integral.

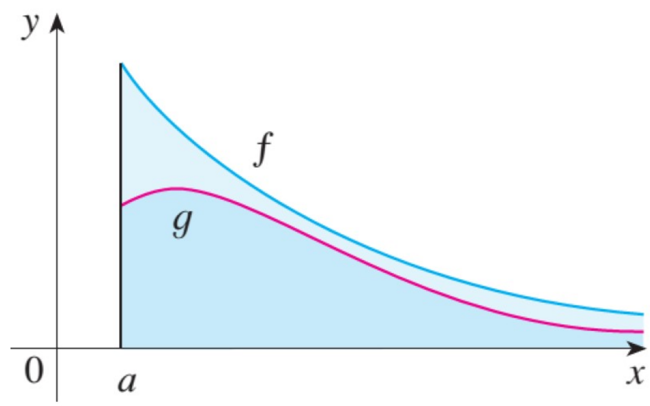
Note: $\int_1^\infty \frac{1}{x}\,dx = \int_1^3 \frac{1}{x}dx + \int_3^\infty \frac{1}{x}dx$. Since $\int_1^3 \frac{1}{x}dx$ is a perfectly ordinary "proper" itegral with value $\ln 3$, the fact that $\int_1^\infty \frac{1}{x}\,dx$ diverges means that $\int_3^\infty \frac{1}{x}\,dx$ also diverges.

We can say that for any $a > 0$, $\int_a^\infty \frac{1}{x^p}\,dx$ $\begin{cases} \underline{\text{converges,}} & \int_a^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} & \text{if } p > 1 \\ \underline{\text{diverges,}} & \int_a^\infty \frac{1}{x^p}\,dx = \infty & \text{if } p \le 1 \end{cases}$

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.



Example For each integral, is it C ("convergent"), D ("divergent"), or U ("not obvious but doable")

$\int_1^\infty \frac{1}{x^2}\,dx$ converges, byt the result above

$\int_1^\infty \frac{1}{3x+x^2}\,dx$ for $x \geq 1$ $x^2 < 3x + x^2$, so

$\frac{1}{x^2} \geq \frac{1}{3x+x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ convergesm $\int_1^\infty \frac{1}{3x+x^2}\,dx$ must also converge

$\int_1^\infty \frac{1}{x^2-\frac{1}{2}}\,dx$ for $x \geq 1$ $\frac{1}{x^2-\frac{1}{2}} \geq \frac{1}{x^2}$

No conclusion about this integral cam be made in comparing it to the convergent integral $\int_1^\infty \frac{1}{x^2}$

$\int_2^\infty \frac{1}{1+\sin^2 x}\,dx$ Since $\sin^2 x \leq 1, \quad 1 + \sin^2 x \leq 2$

so $\frac{1}{1+\sin^2 x} \geq \frac{1}{2}$. Since $\int_2^\infty \frac{1}{2} dx$ diverges, so does the larger integral $\int_2^\infty \frac{1}{1+\sin^2 x} dx$.

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

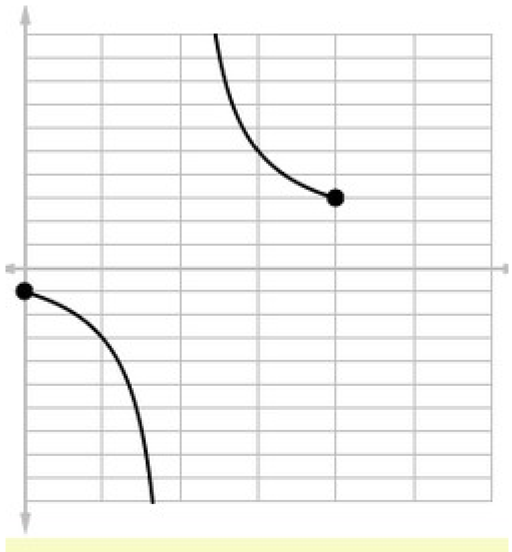
(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example: $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-\frac{1}{2}} = \lim_{t\to 0^+} 2x^{\frac{1}{2}}\Big|_t^1 = \lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$

The integral converges, with value 2

Example $\int_0^4 \frac{1}{x^2-4}\,dx$. The function $f(x) = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$ has an infinite discontinuity (vertical asymptote) at $x = 2$ <u>between</u> 0 and 4. So we break the integral into two parts:



$\int_0^4 \frac{1}{x^2-4}dx \; = \int_0^4 \frac{1}{(x-2)(x+2)}dx = \int_0^2 \frac{1}{(x-2)(x+2)}dx + \int_2^4 \frac{1}{(x-2)(x+2)}dx$

We say that the original integral on the left converges if BOTH parts on the right converge.

Since $\int_0^2 \frac{1}{(x-2)(x+2)}dx = \int_0^2 \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}dx = \frac{1}{4} \lim\limits_{t \to 2^-} \int_0^t \frac{1}{x-2} - \frac{1}{x+2}\, dx$

$\frac{1}{4} \lim\limits_{t \to 2^-} (\ln|x-2| - \ln|x+2|\,) = \frac{1}{4} \lim\limits_{t \to 2^-} \ln\left|\frac{x-2}{x+2}\right| \Big|_0^t$

$= \frac{1}{4}\,(\lim\limits_{t \to 2^-} \underbrace{\ln\left|\frac{t-2}{t+2}\right|}_{\downarrow \atop -\infty} - \ln(1)\,) = \text{d.n.e.}$

Since $\int_0^2 \frac{1}{(x-2)(x+2)}dx$ diverges, we don't even need to look at $\int_2^4 \frac{1}{(x-2)(x+2)}dx$

to conclude that $\int_0^4 \frac{1}{x^2-4}dx$ diverges.