Lecture 26

Reviewed the definition of limit of a sequence, using as first example $\{a_n\}$, where $a_n = \frac{n}{n+1}$

> **1 Definition** A sequence $\{a_n\}$ has the **limit** L and we write
>
> $$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$
>
> if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

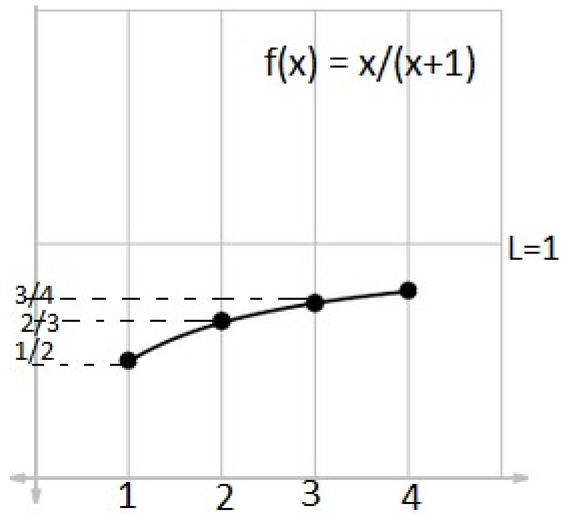
$\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots, \frac{n}{n+1}, \ldots, \frac{10000}{10001}, \ldots \}$

From the pattern, it seems clear that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1,$

Suppose $f(x)$ is a function for which $f(n) = a_n$ for each $n = 1, 2, 3\ldots$ (so that the points $(1, f(1)),\ (2, f(2)),\ \ldots.,\ (n, f(n)),\ \ldots$ lie on the graph of $f(x)$) If $\lim\limits_{x\to\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of $f(x)$: the points $(x, f(x))$ on the graph approach height L as $x \to \infty$. In particular (letting $x = n = 1, 2, 3, \ldots.$) the points $(n, f(n)) = (n, a_n)$ approach height L. So

If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} a_n = L$.

In the example above: let $f(x) = \frac{x}{x+1}$. Since (form Calc I) $\lim\limits_{x\to\infty} \frac{x}{x+1} = \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$, it is also true that $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.



Formal definition of limit of a sequence:

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

In terms of this definition: to prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ we'd say.

Let $\epsilon > 0$ (for example, say $\epsilon = 10^{-6}$.

We want to make $|a_n - 1| = |\frac{n}{n+1} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \epsilon$ for all $n > ?? = N$.

Solving the inequality, we that this will be true if $n + 1 > \frac{1}{\epsilon}$, that is, if $n > \frac{1}{\epsilon} - 1$

(*For a particular example, let* $\epsilon = 10^{-6}$. *Then*

$|a_n - 1| = |\frac{n}{n+1} - 1| < 10^{-6}$ *if* $n > N = \frac{1}{10^{-6}} - 1 = 10^6 + 1$)

Since we have shown that a N can be found for any $\epsilon > 0$ (no matter hoe$w$ small is ϵ), the definition says that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Does the sequence $\{a_n\}$ converge (C) or diverge ? If it converges, what is the limit?

Q1 : $a_n = \frac{(-1)^n\, n}{n+1}$

A) diverges B) converges. limit $= 0$ C) converges,limit $= 1$

D) co$n$verges, limit $= 2$ E) converges, limit $= \sqrt{2}$

<u>Answer</u> Diverges. Informally, $\{a_n\} = \{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}, ... \}$
The a_n's do not approach any particular number L.. (*in fact, the terms* $a_1, a_3, a_5,$ *... get closer and closer to* -1*, but the terms* $a_2, a_4, a_6,$ *... get closer and closer to* 1).

Q2: $a_n = \sqrt{n+1} - \sqrt{n}$

A) diverges B) converges. limit $= 0$ C) converges,limit $= 1$

D) co$n$verges, limit $= 2$ E) converges, limit $= \sqrt{2}$

<u>Answer</u>

We can let $f(x) = \sqrt{x+1} - \sqrt{x}$. Then (as in Calculus I)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x+1}-\sqrt{x}}{1} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0$ (since the numerator is constant and the denominator $\to \infty$)

Of course, we could have just worked with n' s and calculated
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} =$ etc.

Q3: $a_n = \dfrac{\sqrt{2}\,n}{\sqrt{n^2+2^n}}$

A) diverges B) converges. limit $= 0$ C) converges,limit $= 1$

D) co$n$verges, limit $= 2$ E) converges, limit $= \sqrt{2}$

<u>Answer</u> Converges, limit $= 0$.

We can let $f(x) = \dfrac{\sqrt{2}\,x}{\sqrt{x^2+2^x}}$ and calculate (as in Calc I)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2}\,x}{\sqrt{x^2+2x}} = \lim_{x\to\infty} \frac{\sqrt{2}}{(\sqrt{x^2+2^x}\,/\,x)} = \lim_{x\to\infty} \frac{\sqrt{2}}{\sqrt{1+\frac{2^x}{x^2}}}.$$

now, the virtue of shifting to $f(x)$ rather than working with a_n:

$\lim_{x\to\infty} \frac{2^x}{x^2}$ $=$ *(L'Hopital's Rule)* $\lim_{x\to\infty} \frac{2^x\,(ln\,2)}{2x}$ $=$ *(L'Hopital's Rule, again)* $= \lim_{x\to\infty} \frac{2^x\,(ln\,2)^2}{2} = \infty$

So the denominator $\to \infty$ in $\lim_{x\to\infty} \frac{\sqrt{2}}{\sqrt{1+\frac{2^x}{x^2}}}$ so $\lim_{x\to\infty} \frac{\sqrt{2}}{\sqrt{1+\frac{2^x}{x^2}}} = 0$.

Therefore $\lim_{n\to\infty} a_n = 0$.

We discussed two additional items:

<u>Squeeze Theorem for Limits of Sequences:</u>

Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences and

that for each value of n : $a_n \leq b_n \leq c_n$

If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} c_n = L$, then If $\lim\limits_{n\to\infty} b_n = L$

<u>Theorem</u> For a sequence $\{a_n\}$, if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$ also.

Why? Assume that $\lim\limits_{n\to\infty} |a_n| = 0$. By the official definition, this means that for any $\epsilon > 0$, we can find an N such that $|\,|a_n| - 1| < \epsilon$ if $n > N$

But $|\,|a_n| - 0\,| = |\,|a_n|\,| = |a_n| = |a_n - 0|$

So $|a_n - 0| < \epsilon$ if $n > N$ and this is the definition of what it means to say that $\lim\limits_{n\to\infty} a_n = 0$.

<u>Example</u> Find $\lim\limits_{n\to\infty}$ a$_n$, where $a_n = \frac{(-1)^n \sin n}{n}$.

Look first at $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} |\frac{\sin n}{n}|$

Since $0 \leq |\frac{\sin n}{n}| \leq \frac{1}{n}$.

Using $\{a_n\} = \{0, 0, 0, ...)$ (constant sequence) and $\{c_n\} = \{1, \frac{1}{2}, ..., \frac{1}{n}, ...\}$ in the Squeeze Theorem

we conclude that $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} |\frac{\sin n}{n}| = 0$ and therefore (<u>by the Theorem)</u> $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-1)^n \sin n}{n} = 0$ also

Example The preceding theorem only applies **for limit** $= \mathbf{0}$. For example,

(by Q1): $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(-1)^n n}{n+1}$ does not exist, but (first example of lecture)

$$\lim_{n\to\infty} |a_n| = \; = \lim_{n\to\infty} \frac{n}{n+1} = 1$$ (first example of lecture)

We cannot use the theorem to say that because $\lim\limits_{n\to\infty} |a_n| = 1$, then $\lim\limits_{n\to\infty} a_n = 1$ also !