1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

We say that

$\lim\limits_{n\to\infty} a_n = \infty$ if whatever K is chosen, the a_n's are eventually larger than K.
(*<u>more precisely</u>: if for <u>any</u> K, it's possible to find an N such that $a_n > K$ when $n > N$*)

and that

$\lim\limits_{n\to\infty} a_n = -\infty$ if whatever k is chosen, the a$_n$'s are eventually smaller than k.

Assume $a_n \to L$ as $n \to \infty$
and that $f(x)$ is continuous at $x = L$:

then $f(a_n) \to f(L)$ as $n \to \infty$

<u>Example</u> Let $a_n = 2 + \frac{\pi}{2n}$. Then $a_n \to 2$ as $n \to \infty$

Since $f(x) = \sqrt{\sin x}$ is continuous at 2 we can conclude that

$f(a_n) = \sqrt{\sin\left(2 + \frac{\pi}{2n}\right)} \to \sin 2$ as $n \to \infty$

Q1: Suppose $\lim\limits_{n\to\infty} a_n = 11$. What is $\lim\limits_{n\to\infty} a_{n+3}$?

A) 8 **B) 11** C) 14 D) 17 E) d.n.e.

Answer

The original sequence $\{a_n\}$ converges to 11: $a_1, a_2, a_3, a_4, a_5,\ a_7,\ a_8,\ ...\ \longrightarrow 11$

The "new" sequence is the same as the original except that the first 3 terms are missing. For $n = 1$, the first term of $\{a_{n+3}\}$ is a_{3+1}, the second term is a_{3+2}, etc.

So: original sequence $a_1, a_2, a_3, a_4, a_5,\ a_7\ a_{8,...} \longrightarrow 11$
"new" sequence $a_4, a_5,\ a_7\ a_{8,...} \longrightarrow 11$

When a sequence converges to a limit L, this means that the terms of the sequence get closer and closer to L : L is where the terms of the sequence are approaching as $n \to \infty$. The limit of a convergent sequence doesn't change if a finite number of terms at the beginning of the sequence are dropped. For example, above:

$$a_{999},\ a_{1000},\ a_{1001},\ a_{1002},\ ...\ \longrightarrow 11$$

Q2: If r is a constant and $|r| < 1$: what is $\lim\limits_{n\to\infty} \frac{7-7r^n}{1-r}$?

A) r B) 7 C) $\frac{1}{1-r}$ D) $\frac{7}{1-r}$ E) d.n.e.

Since $|r| < 1$, $r^n \to 0$ as $n \to \infty$. (For example, $(\frac{3}{4})^n \to 0$ as $n \to \infty$, and $(-\frac{1}{2})^n \to 0$ as $\to \infty$). So $\lim\limits_{n\to\infty} \frac{7-7r^n}{1-r} = \frac{7-0}{1-r} = \frac{7}{1-r}$.

A sequence $\{a_n)$

is increasing if $a_1 < a_2 < a_3 < ... < a_n < a_{n+1} <$ ..... (that is: $a_n < a_{n+1}$ for every n)

is decreasing if $a_1 > {}_2 > {}_3 > ... > a_n > a_{n+1} >$ ..... (that is: $a_n > a_{n+1}$ for every n)

Example i) The sequence defined by $a_n = (\frac{1}{2})^n$ is obviously decreasing

ii) (using algebra) Show that the sequence defined by $a_n = \frac{n-1}{n}$ is increasing

we must show that $\quad a_n < a_{n+1}$ for every n

we must show that $\quad \frac{n-1}{n} < \frac{(n+1)-1}{n+1} = \frac{n}{n+1}$

But $\quad \frac{n-1}{n} < \frac{n}{n+1}$ is equivalent to

$n^2 - 1 < n^2$, which is rue.

iii) (using a derivative) Is $a_n = \frac{\ln n - 1}{n}$ increasing or decreasing (or neither) ?

Let $f(x) = \frac{\ln x - 1}{x}$.

Then $f'(x) = \frac{x(\frac{1}{x}) - (\ln x - 1)}{x^2} = \frac{2 - \ln x}{x^2} \begin{cases} > 0 & \text{if } x < e^2 \\ = 0 & \text{if } x = e^2 \\ < 0 & \text{if } x > e^2 \end{cases}$

so $f(x)$ is $\begin{cases} \text{increasing} & \text{if } 0 < x < e^2 \\ & (\text{local maximum at } x = e^2) \\ \text{decreasing} & \text{if } x > e^2 \end{cases}$

Since $e^2 \approx 7.39$, this tells us that

$f(1) < f(2 < f(3), f(4) < f(5) < f(6) < f(7)$, then $f(8) > f(9) > f(10) > ...$
that is

$a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7$ then $a_8 > a_9 > a_{10} > ... >$ thereafter

(*can we tell whether $a_7 > a_8$ or $a_7 < a_8$ without further calculation?*)

So the sequence $\{\frac{\ln n - 1}{n}\}$ is neither increasing nor decreasing. But since it is decreasing <u>after the</u> term a_8, we can say that the sequence is "<u>eventually</u> decreasing"

Just for information, here are the first 10 terms of the sequence, rounded to 4 decimal places:

$-1.0000 < -0.1534 < 0.0329 < 0.0966 < 0.1219 < 0.1320 < 0.1351$
$n = 1$ $\qquad\qquad$ $n = 7$

$0.1349 > 0.1330 > 0.1303 > ... > ...$
$n = 8$ $\qquad\qquad$ $n = 10$

A sequence $\{a_n\}$ is

called an upper bound for $\{a_n\}$

$\begin{cases} \text{bounded above if there is a constant } M \text{ (\textit{think big!}) such that every } a_n \leq M \\ \text{bounded below if there is a constant } m \text{ (\textit{think small!}) such that } m \leq \text{ every } a_n \end{cases}$

called a lower bound for $\{a_n\}$

$\{a_n\}$ is bounded if it is both bounded above and below (*so that every a_n is between m and M*)

Example a) Let $a_n = \frac{1}{n} + 1 = \frac{1+n}{n}$

1 is a lower bound for $\{a_n\}$ $-$ because $1 \leq a_n$ for every n
Any number smaller than 1 is also a lower bound. For example, $-\pi$ is a loner bound because $-\pi \leq a_n$ for every n.

1 is the greatest (largest) lower bound for $\{a_n\}$
For example, take any number greater than 1, say $1.001 = \frac{1001}{1000}$. 1.001 is not a lower bound because (just for example) $a_{10000} = 1 + \frac{1}{10000} = 1.0001 < 1.001$.

2 is an upper since every $a_n = \frac{1}{n} + 1 \leq 2$
Any number larger than 2 is also an upper bound. For example, 7 is an upper bound since $a_n \leq 7$ for every n.

2 is the least (smallest) upper bound, because every upper bound must be $\geq a_1 = 2$.

b) Let $a_n = \frac{1+n+\sin n}{n}$

Since $a_n \leq \frac{1+n+\sin n}{n} = \frac{1}{n} + 1 + \frac{\sin n}{n} \leq \frac{1}{n} + 1 + \frac{1}{n} \leq \frac{1}{1} + 1 + \frac{1}{1} = 3$,
3 is an upper bound for $\{a_n\}$.

Since $a_n = \frac{1+n+\sin n}{n} = \frac{1}{n} + 1 + \frac{\sin n}{n} \geq \frac{1}{n} + 1 - \frac{1}{n} \geq 1$,
1 is a lower bound for $\{a_n\}$

Can you identify a least upper bound and greatest lower bound? (Sometimes this is not at all obvious.)

Completeness Property of the Real Numbers (a subtle, rather deep property):

i) an increasing sequence that has an upper bound must have a least upper bound
ii) an decreasing sequence that has a lower bound must have a greatest lower bound.

Monotone Sequence Theorem

a) an increasing sequence that has an upper bound must converge (*in fact, the limit is the least upper bound of the sequence*)

b) a decreasing sequence that has a lower bound must converge (*in fact, the limit is the greatest lower bound of the sequence*)

This theorem guarantees that certain sequences are convergent. Sometimes, when we know that a sequence converges, we can use that fact to find the limit.

Example (similar to one in textbook)

Let $a_1 = 2$ and define $a_{n+1} = \frac{1}{2}(a_n + 8)$

The sequence is bounded above: I claim hat every $a_n < 8$

$a_1 < 8$

Whenever $a_n < 8$, then $a_{n+1} = \frac{1}{2}(a_n + 8) < \frac{1}{2}(8 + 8) = 8$

(*so since we know $a_1 < 8$, it must be that $a_2 < 8$; now, since $a_2 < *$, it must be true that $a_3 < 8$; etc. The "etc." shows why all $a_n < 8$*)

The sequence is increasing:

$$a_1 = 2 \qquad < \qquad a_2 = \frac{1}{2}(a_1 + 8) = 5$$

Whenever $a_n < a_{n+1}$, then
$a_{n+1} = \frac{1}{2}(a_n + 8) < \frac{1}{2}(a_{n+1} + 8) = a_{n+2}$

(*so since we know $a_1 < a_2$, it must be that $a_2 < a_3$; now since $a_2 < a_3$, it must be that $a_3 < a_4$; etc. The etc." shows why $a_n < a_{n+1}$ for all n.*)

By the monotone sequence theorem, there is a number L such that $\lim_{n\to\infty} a_n = L$.
(*Therefore* $\lim_{n\to\infty} a_{n+1} = L$ *also; see Cllicker Q1*)

So $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 8) = \frac{1}{2}(L + 8)$

Therefore $L = \frac{1}{2}(L + 8)$. We can then solve to get $L = 8$.

(*See the example in the textbook, and exercises 79-82 in section 11.1*)