<u>Definition</u> For a series $\sum_{n=1}^{\infty} a_n$. Let $s_n = a_1 + a_2 + \ldots + a_n \ \ (= \sum_{i=1}^{n} a_i)$

We say that $\sum_{n=1}^{\infty} a_n$ <u>converges</u> if $\lim_{n\to\infty} s_n = L$ (*a number*), and write $\sum_{n=1}^{\infty} a_n = L$

We say that $\sum_{n=1}^{\infty} a_n$ <u>diverges</u> if $\lim_{n\to\infty} s_n$ does not exist. (In the case that $\lim_{n\to\infty} s_n$

diverges to $\pm\infty$, we might write $\sum_{n=1}^{\infty} a_n = \infty$ or $-\infty$

Example: $\sum_{n=1}^{\infty} (-2)^n = -2 + 4 - 8 + 16 + \ldots$

$$\begin{aligned} s_1 &= -2 \\ s_2 &= -2 + 4 = 2 \\ s_3 &= -2 + 4 - 8 = -6 \\ s_4 &= -2 + 4 - 8 + 16 = 10 \\ &\vdots \end{aligned}$$

As $n \to \infty$, the partial sums alternate negative and positive, moving further and further from the origin. Clearly, $\lim_{n\to\infty} s_n$ doesn't exist. So $\sum_{n=1}^{\infty} (-2)^n$ <u>diverges</u>

The series $a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots \ = \sum_{n=1}^{\infty} ar^{n-1}$ is called a <u>geometric</u> <u>series</u> and it

$$\begin{cases} \text{converges, with sum } \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

<u>Example</u> $\underset{a_1}{\uparrow a} + \underset{a_2}{\uparrow ar} + ar^2 + \underset{a_4}{\uparrow ar^3} + \ldots$

Suppose we are given that in this geometric series, $a_1 < 0$, $a_2 = 6$, and $a_4 = \frac{2}{3}$

Since $a_2 r^2 = a_4$, we have $6r^2 = \frac{2}{3}$, so $r = \pm\frac{1}{3}$

But we're told that $a_1$negative, Since $a_1 r = a_2 = 6$, r must also be negative. So $r = -\frac{1}{3}$.

Then $a = a_1(-\frac{1}{3}) = 6$, so $a = a_1 = -18$

The series is $-18 + 6 - 2 + \frac{2}{3} - \frac{2}{9} + \ldots$ a geometric series with $r = -\frac{1}{3}$. Since $|r| < 1$, the series converges and its sum is $\frac{a}{1-r} = \frac{-18}{1-(-\frac{1}{3})} = -(18)\frac{3}{4} = -\frac{27}{2}$

Q1: $\frac{90}{100} + \frac{90}{10000} + \frac{90}{1000000} + \cdots = \,?$

A) diverges B) $\frac{99}{100}$ **C)** $\boldsymbol{\frac{90}{99}}$ D) $\frac{999}{1000}$ E) 1

<u>Answer</u> This is a geometric series with $a = \frac{90}{100}$ and ratio $r = \frac{1}{100}$. Since $|\,r\,| < 1$, the series converges and its sum is $\frac{a}{1-r} = \frac{90/100}{1-1/000} = \frac{90}{99}$

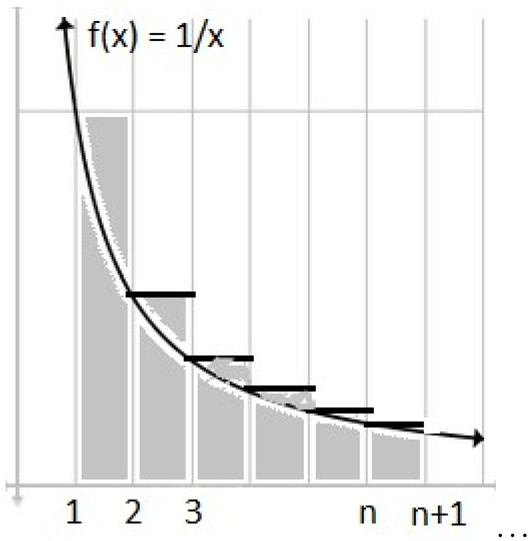
(*Notice that sum of series could also be written as an infinite repeating decimal:*

$\frac{90}{100} + \frac{90}{10000} + \frac{90}{1000000} + \cdots = .90\overline{90}$ ... *(where* $\overline{90}$ *indicates that the "90" continues repeating. You can check on your calculator that* $\frac{90}{99} = 0.909090...$ *to however many digits your calculator displays; the final displayed digit will probably show the effect of rounding: something like* 0.9090901*)*

<u>Example</u> Convert $= 2.0731\overline{31}$ ... to a fraction $\frac{p}{q}$.

$$\begin{aligned}
2.07313131\cdots &= 2.07 \quad + .0031 + .000031 + .00000031 + \cdots \\
&= \frac{207}{100} \quad + \frac{31}{10000} + \frac{31}{1000000} + \frac{31}{100000000} + \cdots \\
&\qquad \textit{geometric series, } a = \frac{31}{10000}\,.\, r = \frac{1}{100} \\
&= \frac{207}{100} \quad + \frac{31/10000}{1-1/100} = \frac{207}{100} + \frac{31}{10000} \cdot \frac{100}{99} \\
&= \frac{207}{100} + \frac{31}{9900} = \frac{20524}{9900}
\end{aligned}$$

Q2: What is the sum of the areas of the shaded rectangles, and how does the sum compare to $\int_1^{n+1} \frac{1}{x} dx$?



A) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}, \quad >$ B) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}, \quad <$

C) $\mathbf{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\boldsymbol{n}}}, \quad >$ D) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad <$

E) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}, \quad =$

<u>Answer</u> The rectangles have heights $1, \frac{1}{2}, \ldots, \frac{1}{n}$ and each has base $= 1$, so the sum of the areas is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. This is larger than the area under the graph of $y = \frac{1}{x}$ over the interval $[1, n+1] = \int_1^{n+1} \frac{1}{x} dx = \ln x \Big|_1^{n+1} = \ln(n+1)$

<u>Example</u> Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?
The n^{th} partial sum $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$ (*see Q2, above*).

So $\lim_{n \to \infty} s_n \; > \; \lim_{n \to \infty} \ln(n+1)$ d.n.e. $(= \infty)$. So $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is called the <u>harmonic series</u>. It diverges because the partial sums $\to \infty$ as you add up more and more terms (that is, as $n \to \infty$). But in this particular example, you'd never "discover" this fact by using a computer to make a table showing $s_1, s_2, \ldots, s_n, \ldots$. For the harmonic series, you need to add up more than 10^{47} terms just to make $s_n > 100$! We'll see how to make an estimate like that later.

Observations

Suppose $\sum_{n=1}^{\infty} a_n$ **converges** (let's say $\sum_{n=1}^{\infty} a_n = L$)

$$
\begin{array}{llcll}
 & & & s_1 & = a_1 \\
s_1 & = a_1 & & s_2 & = a_1 + a_2 \\
s_2 & = a_1 + a_2 & & s_3 & = a_1 + a_2 + a_3 \\
s_3 & = a_1 + a_2 + a_3 & & & \vdots \\
 & \vdots & & & \\
s_{n-1} & = a_1 + a_2 + \cdots + a_{n-1} & & s_n & = a_1 + a_2 + \cdots + a_{n-1} + a_n \\
s_n & = a_1 + a_2 + \cdots + a_{n-1} + a_n & & & \\
\vdots & & & \vdots & \\
\downarrow & & & & \\
L & & & L &
\end{array}
$$

Subtract column 2 $-$ column 1 to get

$$
\begin{array}{lll}
s_1 & = & a_1 \\
s_2 - s_1 = & a_2 & \\
s_3 - s_2 = & a_3 & \\
\vdots & & \\
s_n - s_{n-1} & = & a_n \\
\downarrow & & \downarrow \\
L - L = & 0 &
\end{array}
$$

Conclusion If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Stating this in another equivalent way (in logic, c*alled the contrapositive statement*)

TEST FOR DIVERGENCE

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

Examples

1) $\sum_{n=1}^{\infty} \frac{n^2-2n+5}{2n^2+3n-5}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-2n+5}{2n^2+3n-5} = \frac{1}{2} \neq 0$

2) $\sum_{n=1}^{\infty} (-2)^n$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^n$ d.n.e. (and, for that reason, it's certainly true that $\lim_{n\to\infty} a_n \neq 0$.

CAUTION (digest the following examples!)

$$\begin{cases} \text{For the harmonic series } \sum_{n=1}^{\infty} \frac{1}{n}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \ \underline{\text{BUT}} \text{ the series } \underline{\text{diverges}} \\ \text{For the (geometric) series } \sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0; \text{ the series } \underline{\text{converges}} \end{cases}$$

When $\lim_{n\to\infty} a_n = 0$, the series might converge OR diverge: that is, no conclulsion about the convergence or divergence of the series is possible (without some additional work)

But when $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ must diverge.

When asked whether a series $\sum_{n=1}^{\infty} a_n$ converges or diverges, one of the first few things you should check is:

$$\text{"Is } \lim_{n\to\infty} a = 0 \text{ ?"} \quad \begin{cases} \text{NO} & \text{Series diverges (problem finished, quickly!)} \\ \text{YES} & \text{?? -- need more work to decide about the series} \end{cases}$$