Opening remark:
During a discussion of any infinite series $\sum_{n=1}^{\infty} a_{n}$, there are two different sequences that might come up in the discussion:
i) The sequence of terms for the series: $a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots$ The terms of the series are the numbers we are "adding up"
ii) The sequence of partial sums for the series: $s_{1}, s_{2}, s_{3}, \ldots, s_{n}, \ldots$ where $s_{1}=a_{1}, \quad s_{2}=a_{1}+a_{2}, \ldots, \quad s_{n}=a_{1}+a+\ldots+a_{n}$

Be sure to keep in mind which we are talking about at different times. For example (past lectures):
$\sum_{n=1}^{\infty} a_{n}$ converges (by definition) if $\lim _{n \rightarrow \infty} s_{n}$ exists, that is, if the sequence of partial sums $s_{n}$ has a limit.

The Test for Divergence states that $\sum_{n=1}^{\infty} a_{n}$ diverges if $\lim _{n \rightarrow \infty} a_{n} \neq 0$. This test involves looking at the limit of the sequence of terms,

8 Theorem If $\Sigma a_{n}$ and $\Sigma b_{n}$ are convergent series, then so are the series $\Sigma c a_{n}$ (where $c$ is a constant), $\Sigma\left(a_{n}+b_{n}\right)$, and $\Sigma\left(a_{n}-b_{n}\right)$, and
(i) $\sum_{n=1}^{\infty} c a_{n}=c \sum_{n=1}^{\infty} a_{n}$
(ii) $\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)=\sum_{n=1}^{\infty} a_{n}+\sum_{n=1}^{\infty} b_{n}$
(iii) $\sum_{n=1}^{\infty}\left(a_{n}-b_{n}\right)=\sum_{n=1}^{\infty} a_{n}-\sum_{n=1}^{\infty} b_{n}$

You can check that the preceding statements are true by looking at the partial sums of the series involved.

For example, for (ii) :

$$
\text { suppose } \sum_{n=1}^{\infty} a_{n}=L \text { and } \sum_{n=1}^{\infty} b_{n}=M . \text { Then }
$$

So we know that and therefore can conclude that
partial sums for $\sum_{n=1}^{\infty} a_{n} \quad$ partial sums for $\sum_{n=1}^{\infty} b_{n}$ partial sums for $\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)$
$s_{1}=a_{1}$

$$
t_{1}=b_{1} \quad u_{1}=a_{1}+b_{1}
$$

$s_{2}=a_{1}+a_{2}$
$t_{2}=b_{1}+b_{2}$
$u_{2}=\left(a_{1}+b_{1}\right)+\left(a_{2}+b_{2}\right)$ $=\left(a_{1}+a_{2}\right)+\left(b_{1}+b_{2}\right)$
$\left.s_{n}=a_{1}+\ldots+a_{n}\right)$
$t_{n}=b_{1}+\ldots+b_{n}$
$u_{n}=\left(a_{1}+b_{1}\right)+\ldots+\left(a_{n}+b_{n}\right)$ $=\left(a_{1}+\ldots+a_{n}\right) \ldots+\left(a_{n}+b_{n}\right)$
$\downarrow$
$L$
$\|$
$\sum_{n=1}^{\infty} a_{n}$


$$
\begin{aligned}
& L+M \\
& \sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)
\end{aligned}
$$

So we know that
partial sums for $\sum_{n=1}^{\infty} a_{n}$ partial sums for
$\sum_{n=1}^{\infty} b_{n} \quad$ partial sums for $\sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)$
$s_{1}=a_{1} \quad t_{1}=b_{1} \quad u_{1}=a_{1}+b_{1}$

$$
\begin{array}{ccrl}
s_{2}=a_{1}+a_{2} & t_{2}=b_{1}+b_{2} & u_{2} & =\left(a_{1}+b_{1}\right)+\left(a_{2}+b_{2}\right) \\
\vdots & \vdots & & =\left(a_{1}+a_{2}\right)+\left(b_{1}+b_{2}\right) \\
\vdots & \vdots & & \vdots \\
\left.s_{n}=a_{1}+\ldots+a_{n}\right) & t_{n}=b_{1}+\ldots+b_{n} & u_{n} & =\left(a_{1}+b_{1}\right)+\ldots+\left(a_{n}+b_{n}\right) \\
& & & =\left(a_{1}+\ldots+a_{n}\right) \ldots+\left(a_{n}+b_{n}\right)
\end{array}
$$




$$
\begin{aligned}
& L+M \\
& \| \\
& \sum_{n=1}^{\infty}\left(a_{n}+b_{n}\right)
\end{aligned}
$$

Example: From the preceding lecture, we know that

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{2^{n}}=1 \quad \text { (converges, geometric series) and } \\
& \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1 \quad \text { (converges, telescoping series) }
\end{aligned}
$$

Since both of these converge, we are able to use the properties i), ii), iii) to write
$\sum_{n=1}^{\infty} \frac{3}{2^{n}}+\left(\frac{1}{n}-\frac{1}{n+1}\right)=\sum_{n=1}^{\infty} \frac{3}{2^{n}}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=3 \sum_{n=1}^{\infty} \frac{3}{2^{n}}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=3+1=4$
Notice that we CANNOT write $1=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\sum_{n=1}^{\infty} \frac{1}{n}-\sum_{n=1}^{\infty} \frac{1}{n+1}$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as does $\sum_{n=1}^{\infty} \frac{1}{n+1}$ also).

Example If either one of $\sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=18}^{\infty} a_{n}$ converges, then so does the other because they are connected by the equation: $\sum_{n=1}^{\infty} a_{n}=\sum_{n=1}^{17} a_{n}+\sum_{n=18}^{\infty} a_{n}$. In other words, either $\sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=18}^{\infty} a_{n} \underline{\text { both }}$ converge or both diverge. In particular, if checking whether $\sum_{n=1}^{\infty} a_{n}$ converges it's sufficient to check whether or not $\sum_{n=18}^{\infty} a_{n}$ converges.

Of course the same is true for any $k>1: \sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=k}^{\infty} a_{n} \underline{\text { both }}$ converge or both diverge. Omitting (or changing) a finite number of terms in a series will not change whether or not the series converges or diverges (although, of course, these changes may alter the value of the sum of a convergent series.)

Q1: Answer C ("convergent") or D (divergent) for each infinite series:

$$
\sum_{n=1}^{\infty} \frac{n^{2}+3 n+1}{4 n^{2}-3 n+5}, \quad \sum_{n=3}^{\infty} \frac{e^{n}}{3^{n}}, \quad \sum_{n=0}^{\infty}(-1)^{n+1} \frac{2^{n}}{3^{n+2}}, \quad \sum_{n=4}^{\infty} \frac{5}{n}
$$

A) C, D, C, C
B) C, C, C, D
C) D, D, C, D
D) $\mathbf{D}, \mathrm{C}, \mathrm{C}, \mathrm{D}$
E) C, C, C, D

## Answer D,C,C,D

$\sum_{n=1}^{\infty} \frac{n^{2}+3 n+1}{4 n^{2}-3 n+5}$ diverges because $\lim _{n \rightarrow \infty} \mathrm{a}_{n}=\lim _{n \rightarrow \infty} \frac{n^{2}+3 n+1}{4 n^{2}-3 n+5}=\frac{1}{4}(\neq 0)$ (Test for Divergence)
$\sum_{n=3}^{\infty} \frac{e^{n}}{3^{n}}$ converges: it is a geometric series with $r=\frac{e}{3}<1$
$\sum_{n=0}^{\infty}(-1)^{n+1} \frac{2^{n}}{3^{n+2}}$ converges. It is a geometric series with $r=-\frac{2}{3}$. Since $|r|<1$ the series converges.
$\sum_{n=4}^{\infty} \frac{5}{n}$ diverges: its partial sums are $5 s_{n}$, where $s_{n}$ are the partial sums of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Since $\lim _{n \rightarrow \infty} s_{n}=\infty$, it's also true that $\lim _{n \rightarrow \infty} 5 s_{n}=\infty$.
(More informally: $\sum_{n=1}^{\infty} \frac{1}{n}=\infty$, so $\sum_{N=1}^{\infty} \frac{5}{n}=5 \sum_{n=1}^{\infty} \frac{1}{n}=\infty$ )

Q2: Based on the picture on the left, which inequality is true?

A) $a_{2}+a_{3}+\ldots+a_{n}<\int_{1}^{n} f(x) d x$
B) $a_{2}+a_{3}+\ldots+a_{n}>\int_{1}^{n} f(x) d x$
C) $a_{2}+a_{3}+\ldots+a_{n}>\int_{1}^{n+1} f(x) d x \quad$ D) $a_{1}+a_{2}+\ldots+a_{n-1}>\int_{1}^{n} f(x) d x$
E) $a_{1}+a_{2}+\ldots+a_{n-1}<\int_{1}^{n} f(x) d x$

Answer: in the left picture
sum of the shaded rectangular area $<$ area under the graph over $[1, n]$ $a_{2}+\cdots+a_{n} \quad<\quad \int_{1}^{n} f(x) d x$

The Integral Test For a series $\sum_{n=1}^{\infty} a_{n}$, and that $f(x)$ is a function such that $f(n)=a_{n}$. (Usually, this just means that "the formula for $f(x)$ is the formula for $a_{n}$ with $x$ replacing $n$.")

If on the interval [1. $\infty$ ):

$$
\begin{aligned}
& f(x) \text { is continuous } \\
& f(x)>0 \\
& f(x) \text { is decreasing }
\end{aligned}
$$

Then $\sum_{n=1}^{\infty} a_{n}$ and $\int_{1}^{\infty} f(x) d x$ either both converge OR both diverge.
So under these circumstances what we know about evaluating improper integrals can be used to decide whether an infinite series converges or diverges.
$\underline{\text { Note for the Integral Test: If both } \int_{1}^{\infty} f(x) \text { converges and } \int_{1}^{\infty} f(x) d x=M \text {, then } \sum_{n=1}^{\infty} a_{n}}$ must converge (integral test) but usually the value of the integral is not the same as the sum of the series: $\int_{1}^{\infty} f(x) d x \neq \sum_{n=1}^{\infty} a_{n}$

Example Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge or diverge? Let $f(x)=\frac{1}{\sqrt{x}}$. In this example it is clear that on the interval $[1, \infty)$ the function $f(x)$ is continuous, decreasing and $f(x)>0$. Therefore the integral test can be used.

Since $\int_{1}^{\infty} \frac{1}{\sqrt{x}} d x=\left.\lim _{t \rightarrow \infty} 2 \sqrt{x}\right|_{1} ^{t}$ d.n.e. $(=\infty), \quad \int_{1}^{\infty} \frac{1}{\sqrt{x}} d x$ diverges. The integral test says that we can conclude that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

Why is the integral test true? See detailed discussion in the textbook. Here's an quick summary showing why, when the integral test applies, the series must diverge if the integral does.

From the pictures above:

$f(x)>0$, so the area under the graph $\int_{1}^{n} f(x) d x$ keeps getting larger arn large as $n \rightarrow$ $\infty$. If the integral diverges, then

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \int_{1}^{n} f(x) d x=\infty<\lim _{n \rightarrow \infty}\left(a_{1}+a_{2}+\cdots+a_{n-1}\right)=\lim _{n \rightarrow \infty} s_{n-1} \\
& \text { so } \lim _{n \rightarrow \infty} s_{n-1}=\infty . \text { Therefore } \sum_{n=1}^{\infty} a_{n} \text { diverges. } \\
& \quad \uparrow \\
& \quad(n-1)^{\text {st }} \text { partial sum of series. }
\end{aligned}
$$

The other half of the inequality (from Q1) is used to argue why, if the integral converges, then the series must also converge.

Q3: For what values of $p$ does the series $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ converge?
A) $p>1$
B) $p<-1$
C) $-1<p<1$
D) $p \leq 1$
E) $p \geq 1$

## Answer

- If $p=0, \lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} 1=1 \neq 0$. The Test for Divergence says the series diverges.
- If $p<0$, then $\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} n^{-p}=\infty$ because the exponent $-p>0$. Since $\lim _{n \rightarrow \infty} a_{n} \neq 0$, the series diverges.
- If $p>0$, then $f(x)=\frac{1}{x^{p}}$ is positive, continuous and decreasing on $[1, \infty)$ so the integral test applies. The series converges if $0<p<1$ because for those $p$, the integral $\int_{1}^{\infty} \frac{1}{x^{p}} d x$ diverges, and converges if $p>1$ because for those $p, \int_{1}^{\infty} \frac{1}{x^{p}} d x$ diverges.

To summarize all these cases:
$\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ diverges if $p \leq 1$, converges if $p>1$.
called a " $p$-series"
For example (worked out above), $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty} \frac{1}{n^{1 / 2}}$ diverges.

Example: The integral test works just as well for $\sum_{n=k}^{\infty} a_{n}$ provided $f(x)$ has the properties listed in the test on the interval $[k, \infty)$.

$$
\text { Does } \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text { converge or diverge? }
$$

(notice $n=1$ wouldn't make sense!)
Consider the function $f(x)=\frac{1}{x \ln x}$ for $x$ in $[2, \infty)$

$$
\begin{aligned}
& f(x)>0 \quad(\operatorname{since} \ln x>0 \text { when } x>1) \\
& f(x) \text { is continuous (denominator never } 0 \text { for } x \geq 2) \\
& f(x) \text { is decreasing on the interval }[2 . \infty):
\end{aligned}
$$

This is clear from the formula but a way to check "decreasing?" that can be useful in more complicated cases is to check whether $f^{\prime}(x)<0$. Here $f^{\prime}(x)=\frac{-(\ln x+1)}{(x \ln x)^{2}}$. Since the denominator is positive on $[2, \infty)$, we need to think about the numerator:

$$
\begin{array}{r}
-\ln x-1<0 \text { is equivalent to } \quad \begin{aligned}
\ln x & +1>0 \\
\ln x & >-1
\end{aligned} \\
e^{\ln x}>e^{-1} \\
x>\frac{1}{e} \approx 0.37
\end{array}
$$

So, for $x$ in $[2, \infty)$ the numerator is $<0$, and therefore $f^{\prime}(x)<0$

$$
(f(x) \text { decreasing) on }[2, \infty)
$$

So the integral test applies to $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.
Consider $\int_{2}^{\infty} \frac{1}{x \ln x} d x$ :

$$
\int \frac{1}{x \ln x} d x=(\text { let } u=\ln x)=\int \frac{1}{u} d u=\ln |u|+C=\ln (|\ln x|)+C
$$

Since $\int_{2}^{\infty} \frac{1}{x \ln x} d x=\lim _{t \rightarrow \infty} \int_{2}^{t} \frac{1}{x \ln x} d x=\left.\lim _{t \rightarrow \infty} \ln (|\ln x|)\right|_{2} ^{t}=\lim _{t \rightarrow \infty} \ln (|\ln | t \mid)-\ln (\ln 2)=\infty$, the integral diverges and therefore $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

