Example The series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the integral test. Suppose $\sum_{n=1}^{\infty} \frac{1}{n^4} = s$. We can't find s exactly with what we know, but we can approximate s using one of the partial sums s_n. How large should n be so that the approximation error $s - s_n < 10^{-6}$?

In the last lecture, we learned that (in the integral test) $s - s_n = R_n < \int_n^{\infty} f(x)\,dx$

So, in this example, we want to choose n so that

we want

$$s - s_n = R_n < \int_n^{\infty} \frac{1}{x^4} dx = \lim_{t\to\infty} -\frac{1}{3x^3}\Big|_n^{\infty} = \frac{1}{3n^3} < 10^{-6}$$

so we need $3n^3 > 10^6$, that is $n > \sqrt[3]{\frac{10^6}{3}} \approx 69.3$, so $n = 70$ will work.

$s - s_{70} < 10^{-6}$, and (at least with enough work, or a computers help) we can actually compute s_{70}.

From the last lecture:

Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series where all $a_n > 0$ and all $b_n > 0$ Suppose $a_n \leq b_n$ for all n. Then

- if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges
- if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

Note: it's not necessary that the series begin with $n = 1$. The comparison test works equally well if the series are $\sum_{n=k}^{\infty} a_n$ and $\sum_{n=k}^{\infty} b_n$

Q1: You can decide that $\sum\limits_{n=7}^{\infty} \frac{1}{n-3}$ diverges by comparing it to which of the following series?

A) $\sum\limits_{n=7}^{\infty} \frac{1}{n-4}$ B) $\sum\limits_{n=7}^{\infty} \frac{1}{n-2}$ C) $\sum\limits_{n=7}^{\infty} \frac{1}{n-1}$ **D)** $\boldsymbol{\sum\limits_{n=7}^{\infty} \frac{1}{n}}$ E) $\sum\limits_{n=7}^{\infty} \frac{1}{n+1}$

Answer for $n > 7:$ $n - 3 < n$, so $\frac{1}{n-3} > \frac{1}{n}$. Since $\sum\limits_{n=7}^{\infty} \frac{1}{n}$diverges $(= \infty)$, the larger series $\sum\limits_{n=7}^{\infty} \frac{1}{n-3}$ also diverges by the comparison test.

Example For which the following two series can we decide convergence or divergence by a comparison to the series $\sum\limits_{n=2}^{\infty} \frac{1}{7^n}$?

$$\sum_{n=2}^{\infty} \frac{1}{7^n+2n} \qquad\qquad \sum_{n=2}^{\infty} \frac{1}{7^n-2n}$$

$$7^n + 2n > 7^n \qquad\qquad 7^n - 2n < 7^n$$

so $\frac{1}{7^n+2n} < \frac{1}{7^n}$ $\qquad$ $\frac{1q}{7^n-2n} < \frac{1}{7^n}$

so $\sum\limits_{n=2}^{\infty} \frac{1}{7^n+2n} \leq \sum\limits_{n=2}^{\infty} \frac{1}{7^n}$ $\qquad$ $\sum\limits_{n=2}^{\infty} \frac{1}{7^n-2n} \geq \sum\limits_{n=2}^{\infty} \frac{1}{7^n}$

Since $\sum\limits_{n=2}^{\infty} \frac{1}{7^n}$ converges (it's a geometric series, $r = \frac{1}{7}$) the smaller series $\sum\limits_{n=2}^{\infty} \frac{1}{7^n+2n}$ must also converge by the comparison test; no conclusion about $\sum\limits_{n=2}^{\infty} \frac{1}{7^n-2n}$is possible from the second comparison.

Our intuition should tell us, however, that for very large n, 7^n is <u>much larger than</u> $2n$, so that $\frac{1}{7^n-2n}$ isn't much different from $\frac{1}{7^n}$. So we might "guess" that $\sum\limits_{n=2}^{\infty} \frac{1}{7^n-2n}$ also converges – perhaps some other comparison would work. We'll decide about this series later in the lecture.

Example Does $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ converge or diverge?

Thinking of other series, we know that when p is constant, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$. For example, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. But when n is "large enough" $\ln n$ will be > 2 – so, when n is "large enough" $n^{\ln n} > n^2$ and $\frac{1}{n^{\ln n}} < \frac{1}{n^2}$. So we try to compare $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ to compare $\sum_{n=1}^{\infty} \frac{1}{n^2}$ more carefully:

$\ln e^2 = 2$ and $e^2 \approx 7.39$, so $\ln n > 2$ when $n \geq 8$

So when $n \geq 8$: $n^{\ln n} > n^2$ and $\frac{1}{n^{\ln n}} < \frac{1}{n^2}$.

Therefore $\sum_{n=8}^{\infty} \frac{1}{n^{\ln n}} \leq \sum_{n=8}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=8}^{\infty} \frac{1}{n^2}$, so does the smaller series $\sum_{n=8}^{\infty} \frac{1}{n^{\ln n}}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}} = \sum_{n=1}^{7} \frac{1}{n^{\ln n}} + \sum_{n=8}^{\infty} \frac{1}{n^{\ln n}}$, $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ must converge.

Given $\sum_{n=1}^{\infty} a_n$ (with $a_n > 0$), the limit comparison test often is easier to use than the comparison test – usually there is no need to manipulate inequalities – although you still need to guess/pick a series $\sum_{n=1}^{\infty} b_n$ whose behavior you know to compare to $\sum_{n=1}^{\infty} a_n$.

The Limit Comparison Test

Suppose a_n and $b_n > 0$ for all n. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

If $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ where $L \neq 0$, $L \neq \infty$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

A justification is given in the textbook and was given in the lecture. Again, it doesn't matter whether the two series begin with $n = 1$ or not.

Example Earlier, we guessed that $\sum_{n=2}^{\infty} \frac{1}{7^n - 2n}$ probably behaves the same as $\sum_{n=2}^{\infty} \frac{1}{7^n}$ (converges). We can use the limit comparison test to justify this:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(7^n - 2n)}{1/7^n} = \lim_{n\to\infty} \frac{7^n}{7^n - 2n} = \lim_{n\to\infty} \frac{1}{1 - \frac{2n}{7^n}} = 1.$$

By the limit comparison test, $\sum_{n=2}^{\infty} \frac{1}{7^n - 2n}$ converges because $\sum_{n=2}^{\infty} \frac{1}{7^n}$ converges.

Q2: Using the Limit Comparison Test, we can decide that both of the following series have the same behavior: converge or diverge?

$$\sum_{n=2}^{\infty} \frac{1}{e^n + 5n + 1} \qquad \sum_{n=2}^{\infty} \frac{1}{e^n - 1}$$

A) Both Converge B) Both diverge

Answer Use the limit comparison test:

$$\lim_{n\to\infty} \frac{1/(e^n - 1)}{1/(e^n + 5n + 1)} = \lim_{n\to\infty} \frac{e^n + 5n + 1}{e^n - 1} = \lim_{n\to\infty} \frac{1 + \frac{5n}{e^n} + \frac{1}{e^n}}{1 - \frac{1}{e^n}} = 1.$$

So both series diverge or both series converge.

If we use the limit comparison test again, we can see that $\sum_{n=2}^{\infty} \frac{1}{e^n - 1}$ and $\sum_{n=2}^{\infty} \frac{1}{e^n}$ both converge or both diverge: $\lim_{n\to\infty} \frac{1/(e^n - 1)}{1/e^n} = \lim_{n\to\infty} \frac{e^n}{e^n - 1}$

$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{e^n}} = 1$. Since $\sum_{n=2}^{\infty} \frac{1}{e^n}$ is a geometric series with $r = \frac{1}{e} < 1$, it converges.

Therefore $\sum_{n=2}^{\infty} \frac{1}{e^n + 5n + 1}$ and $\sum_{n=2}^{\infty} \frac{1}{e^n - 1}$ both converge.