We reviewed the comparison test and the limit comparison test. Note that these tests, and the earlier integral test, only are valid for series with positive terms a_n

Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series where all $a_n > 0$ and all $b_n > 0$

Suppose $a_n \le b_n$ for all n. Then

- if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges
- if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

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The Limit Comparison Test

Suppose a_n and $b_n > 0$ for all n. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L \neq 0$. $L \neq \infty$

then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

Q1: (Comparison Test): The series $\sum_{n=1}^{\infty} \frac{1}{n!}$

A) $\sum_{n=1}^{\infty} \frac{1}{n!}$ diverges because $\frac{1}{n!} > \frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ diverges

B) $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{1}{n!} > \frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges

C) $\sum_{n=1}^{\infty} \frac{1}{n!}$ diverges because $\frac{1}{n!} < \frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges

D) $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n!}}$ converges because $\boldsymbol{\frac{1}{n!} < \frac{1}{2^{n-1}}}$ and $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}}$ converges

E) $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{1}{n!} < \frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ diverges

<u>Answer</u> $\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n} = \frac{1 \cdot 1 \cdot 1 \cdot \qquad \cdots \cdot 1 \cdot \quad 1}{1 \cdot 2 \cdot 3 \cdot \cdots \quad (n-1) \cdot n}$

$$< \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^{n-1}}$$

So $\sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (geometric series, $r = \frac{1}{2}$).

$\sum_{n=1}^{\infty} \frac{1}{n!}$ must also converge by the Comparison Test.

Q2: You can use the Limit Comparison Test to decide that $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

A) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ diverges by using the Limit Comparison Test with the series $\sum_{n=2}^{\infty} \frac{1}{n}$

B) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by using the Limit Comparison Test with the series $\sum_{n=2}^{\infty} \frac{1}{n}$

C) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ diverges by using the Limit Comparison Test with the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

D) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by using the Limit Comparison Test with the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

E) $\boldsymbol{\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}}$ converges by using the Limit Comparison Test with the series $\boldsymbol{\sum_{n=2}^{\infty} \frac{1}{n^2}}$

For very large n, the terms $a_n = \frac{1}{n\sqrt{n^2-1}}$ are approximately the same as $\frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$, so I'm guessing that $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ behaves like $\sum_{n=2}^{\infty} \frac{1}{n^2}$. To check out this guess, I use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1/(n\sqrt{n^2-1})}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{(\sqrt{n^2-1})/n}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1-1/n^2}} = 1.$$

By the Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges because $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges.

Example: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \quad \textbf{(for } n \geq 3\textbf{, each red fraction is } \leq 1\textbf{)}$$

$$\leq \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots \quad 1 \cdot 1 = \frac{2}{n^2}$$

Since $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges, $\sum_{n=3}^{\infty} \frac{n!}{n^n}$ converges by the Comparison Test, so $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ also converges.

Note I could have written

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \quad \textbf{(for } n \geq 2\textbf{, each blue fraction is } \leq 1\textbf{)}$$

$$\leq \frac{1}{n} \cdot 1 \cdot 1 \cdot 1 \cdots \quad 1 \cdot 1 = \frac{1}{n}$$

This would then tell me that $\sum_{n=2}^{\infty} \frac{n!}{n^n} \leq \sum_{n=2}^{\infty} \frac{1}{n}$ which is true but not a useful comparison since the larger series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

We now start looking at some tests for series where not all the terms are positive. The simplest kind of series with "mixed signs" is an alternating series:

$$b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (*)$$

$$\text{(or)} \qquad -b + b_2 - b_3 + b_4 - b_5 + b_6 - \cdots \qquad (**)$$

Here, we are assuming that the b_n's themselves are all positive, so that the "+" and " $-$ " signs cause the signs to alternate (*For example, if* b_2 *were itself negative, then* $-b_2$ *in (*) would positive and there would be no alternation of signs between the first term and the second.*)

<u>Alternating Series Test</u>

Suppose all b_n's > 0

$\lim_{n\to\infty} b_n = 0$

<u>and</u> b_n is a decreasing sequence (each $b_n > b_{n+1}$)

Then the <u>alternating series</u> $b_1 - b_2 + b_3 - b_4 + \cdots$ (*)

<u>or</u> $-b_1 + b_2 - b_3 + b_4 - \cdots$ (**)

<u>converges</u>.

If an alternating series converges and has sum s, then we can approximate s with a partial sum s_n and the error $s - s_n$ satisfies

$$|s - s_n| < b_{n+1} \text{ (} = \text{ the magnitude of the first "ignored" term in the sum)}$$

<u>Example</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

Here, $b_n = \frac{1}{n}$. Since $b_n > 0$,

$\lim_{n\to\infty} b_n = 0$ and

$b_n > b_{n+1}$

the Altenating Series Test applies and the series converges. Let s be its sum.

If we approximate s by the partial sum s_7, then $|s - s_7| < b_8 = \frac{1}{8}$

(*By methods found near the end of Chapter 11, it tuens out that the exact sum* $s = \mathit{ln}\, 2$.
So $|s - s_7| = |s - (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7})|$

$= |\mathit{ln}\, 2 - (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7})| \leq \frac{1}{8}.$)

Example Does the Alternating Serires Test apply to $\sum_{n=1}^{\infty}(-1)^{2n}\frac{n}{n+1}$?

No. Since $(-1)^{2n}=1$ for every n, $\sum_{n=1}^{\infty}(-1)^{2n}\frac{n}{n+1}=\sum_{n=1}^{\infty}\frac{n}{n+1}$ (*which diverges*).
$\sum_{n=1}^{\infty}(-1)^{2n}\frac{n}{n+1}$ is not an alternating series: all terms are positive !

Example Does the Alternating Series Test apply to $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{3n^2+5}$?
No. Here $b_n=\frac{n^2}{3n^2+5}$ and $\lim_{n\to\infty}\frac{n^2}{3n^2+5}=\frac{1}{3}\ (\neq 0)$.
Note: if we write $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{3n^2+5}=\sum_{n=1}^{\infty}a_n$, where $a_n=(-1)^{n+1}\frac{n^2}{3n^2+5}$, then $\lim_{n\to\infty}a_n\neq 0$ (this limit doesn't even exist!), so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{3n^2+5}$ divegres (by the Test for Divergence).