Theorem If $\sum |a_n|$ converges, then $\sum a_n$ must also converge.

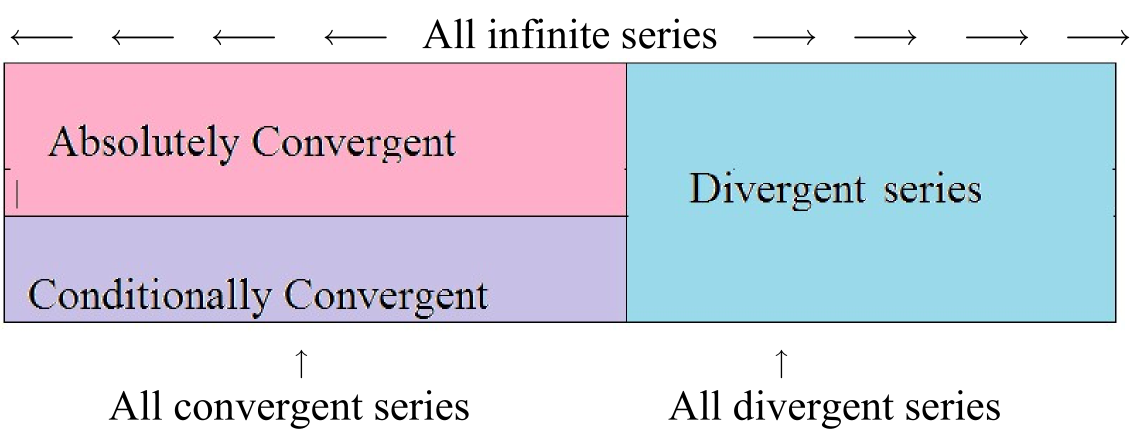
(We say then that

$\sum a_n$ converges absolutely.)

If $\sum a_n$ converges but $\sum |a_n|$ does not converge, then we say that

$\sum a_n$ converges conditionally

The box is the "universe" of all infinite series. It is divided into two parts: the convergent series and the divergent series. The convergent series are subdivided into two kinds, those that are absolutely convergent and those that are only conditionally convergent.



Example (from last lecture)

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{ converges (Alternating Series Test)}$$

but $\displaystyle\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

so $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ converges conditionally, but not absolutely.

The Ratio Test For a series $\sum a_n$ (where the a_n's may/may not have mixed signs):

$$\text{If } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L, \text{ then } \begin{cases} \text{if } L<1 & \text{then } \sum a_n \text{ converges } \underline{\text{absolutely}} \\ \text{if } L>1 \text{ or } =\infty & \text{then } \sum a_n \text{ diverges} \\ \text{if } L=1 & \text{ratio test } \underline{\text{fails}}\text{: series behavior is undecided} \\ & (\textit{must try some other test}) \end{cases}$$

Q1: For the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} 6 & \text{when } n \text{ is odd} \\ -3 & \text{when } n \text{ is even} \end{cases}$

What can you say about $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$?

A) limit does not exist and ratio test fails

B) limit $=1$ and the ratio test says the series converges absolutely

C) limit $=2$ and ratio test says the series diverges

D) limit $<\frac{1}{2}$ and the ratio test says the series converges absolutely

E) limit $=1$ and the ratio test fails

Answer: Look at the $\left|\frac{a_{n+1}}{a_n}\right|$ for several values of n :

$$n=1: \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_2}{a_1}\right| = \left|\frac{-3}{6}\right| = \frac{1}{2}$$

$$n=2: \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_3}{a_2}\right| = \left|\frac{6}{-3}\right| = 2$$

$$n=3: \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_4}{a_3}\right| = \left|\frac{-3}{6}\right| = \frac{1}{2}$$

$$n=4: \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a_5}{a_4}\right| = \left|\frac{6}{-3}\right| = 2$$

$$\vdots \qquad\qquad\qquad \vdots$$

The sequence $\left|\frac{a_{n+1}}{a_n}\right|$ $s:\ \frac{1}{2},\ 2,\ \frac{1}{2},\ 2,\ \frac{1}{2},\ 2,\ \cdots$ So $\lim_{nj\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exiist. There is no "L" and that Ratio Test fails (cannot be used)

We discussed in the lecture why the Ratio Test is true when $L < 1$. This is presented also in the textbook, together with the case $L > 1$.

Example Trying the Ratio Test on $\sum_{n=1}^{\infty} \frac{1}{n}$ gives $L = \lim_{n \to \infty} \frac{1/(n+1)}{1/n} = \lim_{n \to \infty} \frac{n}{n+1} = 1$

Trying the Ratio Test on $\sum_{n=1}^{\infty} \frac{1}{n^2}$ gives $L = \lim_{n \to \infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1$

In both cases, $L = 1$, but we already know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

This example illustrates that the series could "go either way" – that is, the Ratio Test itself can't tell us, when $L = 1$, whether the series will converge or diverge.

Q2: What does the Ratio Test tell you about the x's for which $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n\,5^n}$ converges?.

A) Series converges absolutely for all x

B) Series converges absolutely when $|x| < 5$

C) Series converges absolutely when $|x| < 1$

D) Series converges absolutely when $|x| < \frac{1}{5}$

E) Series converges absolutely only when $x = 0$.

Answer: $\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\frac{x^{n+1}}{(n+1)\,5^{n+1}}}{(-1)^{n-1}\frac{x^n}{n\,5^n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\,5^{n+1}}\cdot\frac{n\,5^n}{x^n}\right|$

$= \lim_{n\to\infty}\left|\frac{n\,x}{(n+1)\,5}\right| = \frac{|x|}{5}$. The Ratio Test says that the series converges absolutely when $L = \frac{|x|}{5} < 1$, that is, for $-5 < x < 5$.

Continuing with the series in Q2: the Ratio Test also says that this series diverges when $L = \frac{|x|}{5} > 1$, that is, when $x > 5$ or $x < -5$.

The Ratio Test fails when $L = \frac{|x|}{5} = 1$, that is, when $x = \pm 5$. So, if we want to know what happens for those x, we must test the series in some other way:

When $x = 5$: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n\,5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{5^n}{n\,5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$
$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which converges, by the Alternating Series Test. (*Note: at $x = 5$, the convergence is conditional, not absolute because* $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{5^n}{n\,5^n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

When $x = -5$: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-5)^n}{n\,5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n 5^n}{n\,5^n} = \sum_{n=1}^{\infty}(-1)^{2n-1}\frac{1}{n}$. Since the exponent $2n-1$ is always odd, $(-1)^{2n-1} = -1$ for every n. Therefore $\sum_{n=1}^{\infty}(-1)^{2n-1}\frac{1}{n}$
$= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots = -(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots)$ which diverges.

To sum up: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n\,5^n}\begin{cases}\text{converges for any } x \text{ in } (-5,\ 5] \\ \text{diverges for } x \text{ in } (-\infty, 5] \cup (5, \infty)\end{cases}$

In the lecture, we stated the Root Test. In the textbook, the proof is relegated to an exercise. We will look at the proof of one part of the Root Test in the next lecture.

<u>The Root Test</u> For a series $\sum a_n$ (where the a_n's may/may not have mixed signs):

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L, \text{ then } \begin{cases} \text{if } L < 1 & \text{then } \sum a_n \text{ converges } \underline{\text{absolutely}} \\ \text{if } L > 1 \text{ or } = \infty & \text{then } \sum a_n \text{ diverges} \\ \text{if } L = 1 & \text{root test } \underline{\text{fails}}\text{: series behavior is undecided} \\ & (\textit{must try some other test}) \end{cases}$$

Note that the conclusions that you draw from L are the same conclusion as for the Ratio Test.

Q3: What does the root test tell you about the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$?

A) $L < 1$ so the series converges absolutely

B) $L = 1$ and the root test fails

C) $L > 1$ so the series diverges

D) $L < 1$ so the series converges absolutely, and therefore <u>also</u> converges contitionally

<u>Answer</u> For $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$, we have $a_n = \frac{(-2)^n}{n^n}$, so $\lim_{n\to\infty} \sqrt[n]{|\,a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|}$

$= \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{2}{n} = 0$. So, by the Root Test, the series converges absolutely.