We reviewed the ratio and root tests.

The Ratio Test For a series $\sum a_n$ (where the a_n's may/may not have mixed signs):

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\begin{cases} \text{if } L < 1 & \text{then } \sum a_n \text{ converges absolutely} \\ \text{if } L > 1 \text{ or } = \infty & \text{then } \sum a_n \text{ diverges} \\ \text{if } L = 1 & \text{ratio test fails: series behavior is undecided} \\ & (\textit{must try some other test}) \end{cases}$

The Root Test For a series $\sum a_n$ (where the a_n's may/may not have mixed signs):

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then $\begin{cases} \text{if } L < 1 & \text{then } \sum a_n \text{ converges absolutely} \\ \text{if } L > 1 \text{ or } = \infty & \text{then } \sum a_n \text{ diverges} \\ \text{if } L = 1 & \text{root test fails: series behavior is undecided} \\ & (\textit{must try some other test}) \end{cases}$

Note that the conclusions that you draw from L are the same for both the Ratio Test and the Root Test.

Why does the Root Test work? Here's the explanation for the case when $L > 1$:

Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$. Pick a number r, where $1 < r < L$.

Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, the values of $\sqrt[n]{|a_n|}$ get as closed as desired to L when n is large enough. So we can pick a number N such that, when $n \geq N$, $\sqrt[n]{|a_n|} > r$.

Then, when $n \geq N$, $|a_n| > r^n$. Since $r > 1$, $r^n \to \infty$ as $n \to \infty$, and so $\lim_{n\to\infty} |a_n| = \infty$ also. Therefore $\lim_{n\to\infty} a_n \neq 0$ By the Test for Divergence, $\sum a_n$ must diverge.

The justification for the Root Test when $L < 1$ is similar to the justification we gave in class when $L < 1$ for the Ratio Test, Q1 Apply the root test to the series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^n}$. What conclusion do you draw?

A) The series diverges

B) The series converges absolutely

C) The series converges conditionally

D) The Root Test is inconclusive

Answer $a_n = \frac{(n!)^n}{n^n}$, so $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{n^n}} = \lim\limits_{n\to\infty} \frac{n!}{n} = \lim\limits_{n\to\infty} (n-1)! = \infty$
so, by the Root Test, the series diverges.

Suppose $c_0,\ c_1, ..., c_n, ...$ are constants. A series like

(*) $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ ...\ + c_n x^n + \ ...$ is an example of a power series

Informally, you can imagine it as a "infinitely long polynomial." It will converge when certain x values (for example $x = 0$) are substituted in, and may diverge when certain other x values are substituted into (*).

For those x's that make the series converge, the series has a sum which depends on the x value:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ ...\ + c_n x^n + \ ... \quad = f(x)$$

For example, $f(0) = c_0 + c_1(0) + c_2(0)^2 + ... = c_0$. The domain for the function defined by the equation (*) is the set of x's that make the series converge.

Example: $1 - x + x^2 - x^3 + ... = f(x)$

This is a geometric series where the first term $a = 1$ and the ratio $r = -x$. Therefore it converges when $|r| = |-x| = |x| = 1$, that is, when $-1 < x < 1$ so

(**) $1 - x + x^2 - x^3 + ... = \frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x} = f(x), \quad -1 < x < 1$

Note: The formula $\frac{1}{1+x}$ makes sense for all $x \neq -1$, But only for $-1 < x < 1$ does the sum of the power series "add up" to $f(x) = \frac{1}{1+x}$. If we think of $f(x)$ as being defined as the sum of the series, then $f(x)$ is only defined where the series "adds up" to something: $-1 < x < 1$.

We could also think of “reading” equation (**) from right to left: <u>given $\frac{1}{1+x}$</u>
<u>can we think of a series</u> whose sum (at least for certain x's) is $\frac{1}{1+x}$. We can reason that $\frac{1}{1+x}$ looks like $\frac{a}{1-r}$ = the sum of a geometric series with first term z and ratio r if we choose $a = 1$ and $r = -x$. Then we could write down the series $1 - x + x^2 - x^3 + ...$ as a power series whose sum is $\frac{1}{1+x}$, provided that $|r| = |-x| = |x| < 1$.

<u>Example</u>

We can create new series and functions from equation (**).

- Substitute $x = 3w$ into (**).

$$1 - 3w + 9w^2 - 27w^3 + \ ... \ = \ \frac{1}{1+3w}$$

But to use (**), we need $|x| = |3w| < 1$, so the new equation is valid if $|w| < \frac{1}{3}$, that is $-\frac{1}{3} < w < \frac{1}{3}$.

- For $-1 < x < 1$, multiply (**) by $2x^2$ to get

$$2x^2 - 2x^3 + 2x^4 - \ ... \ = \ \frac{2x^2}{1+x} \qquad \text{for } -1 < x < 1.$$

Again (reading right to left) we could have asked: “find a series that represents the function $f(x) = \frac{2x^2}{1+x}$. This function looks like the sum of a geometric series $\frac{a}{1-r}$ where $a = 2x^2$ and $r = -x$. We could then write down the series on the left, with the restriction that $|r| = |-x| = |x| < 1$.

- In (**), substitute $x = \sin\theta$ to get

$$1 - \sin\theta + \sin^2\theta - \sin^3\theta + \ ... \ = \ \frac{1}{1+\sin\theta} \quad \text{provided } |x| = |\sin\theta| < 1,$$

that is, when $\sin\theta \neq \pm 1$. Of course when $\sin\theta = 1$, $\frac{1}{1+\sin\theta} = \frac{1}{2}$, but the infinite series doesn't converge (so the series doesn't add to $\frac{1}{2}$).
(*Notice, though, that* $1 - \sin\theta + \sin^2\theta - \sin^3\theta + \ ...$ *is NOT a power series because the series is not a sum of powers of the variable θ: θ, θ^2, θ^3, etc.*

The series $\sum_{n=0}^{\infty} c_n(x-a)^n \;\; = c_0 + c_1(x-a) + c_2(x-a)^3 + ...$
is called a power series centered at $a.$

(The previous series $1 - x + x^2 - x^3 + ...$ is a power series centered at $a = 0$)

Notice that when $x = a$, the series must converge:

$$\sum_{n=0}^{\infty} c_n(a-a)^n \;\; = c_0 + c_1(a-a) + c_2(a-a)^3 + ... \;\; = \; c_0$$

The following theorem indicates why we say that $\sum_{n=0}^{\infty} c_n(x-a)^n$ is centered at a :

Theorem For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following must be true:

1) The series converges only for $x = a$

2) The series converges for all x

3) There is a number $R > 0$ such that
- series converges when $|x-a| < R$ $\qquad (a - R \; < x \; < a + R)$

and
- series diverges when $|x-a| > R.$

So in case 3) the series converges in a interval centered at a and with radius $R.$ R is called the radius of convergence of the power series. (*The same is true in case 2) if we say that* $R = \infty$*; and also in case 1) if we say* $R = 0$)

In case 3), the series might either converge or might diverge when $|x-a| = R$*, that is, when* $x = a - R =$ *left endpoint of the interval, or* $x = a + R =$ *right endpoint of the interval.*

The proof that these are always the only possibilities is a bit tricky (*see Appendix in textbook if you're interested, or talk with me.*) But in most cases, the ratio or root test will work to show that 1), 2) or 3) must be true for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$

Q2 For what x's does the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

A) Only for $x = 0$ B) For $-1 < x < 1$ C) For $-1 \leq x \leq 1$
nD) For $-4 \leq x \leq 4$ E) For all x\\

Answer Use the Ratio Test : $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| |x| = \lim_{n\to\infty} \frac{1}{n+1} |x|.$
But for any particular value of x, $\lim_{n\to\infty} \frac{1}{n+1} |x| = 0 < 1.$ So, by the Ratio Test, the series converges to some function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x.$ (So, Case 2 in the Theorem: $R =$ radius of convergence $= \infty$).