An infinite number of mathematicians walk into a bar. The first one orders a beer. The second orders half a beer. The third, a quarter of a beer. The bartender says "You're all idiots", and pours two beers.

An infinite number of mathematicians walk into a bar. The first one orders a beer. The second orders half a beer. The third orders a third of a beer. The bartender bellows, "Get the hell out of here, are you trying to ruin me?"

For x in the interval of convergence :

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots \qquad = f(x)$$

we can define a function $f(x)$ (in a "fancy" way) by writing a power series. In calculus, when we have a function, we are interested in whether we can find its derivative or antiderivative. The following theorem tells us that this is easy <u>for a power series</u>.

Theorem Suppose the power series $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$ has radius of convergence $R > 0$. <u>For x in the interval</u> $(a-R,\ a+r)$ we can define

$$\boldsymbol{f(x) \;\; = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots}$$

Then

- $$\boldsymbol{f'(x) \;\; = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1} = c_1(x-a) + 2c_2(x-a)^2 + 3c_3(x-a)^2 + \ldots}$$

•• $$\boldsymbol{\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \tfrac{(x-a)^{n+1}}{n+1} + C = c_0(x-a) + \tfrac{c_1}{2}(x-a)^2 + \tfrac{c_2}{3}(x-a)^3 + \ldots + C}$$

That is, <u>inside the interval of convergence, a power series can be integrated or differentiated</u> "<u>term by term</u>" just as if it were a polynomial.

At the endpoints of the interval, where $x = a \pm R$, the formulas • and •• may not be true, <u>even if</u> the power series (*) converges at the endpoint.

Q1. Suppose $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} + \ldots \quad (|x| < 2)$

Which of the following is $f'(1)$ and what is its value?

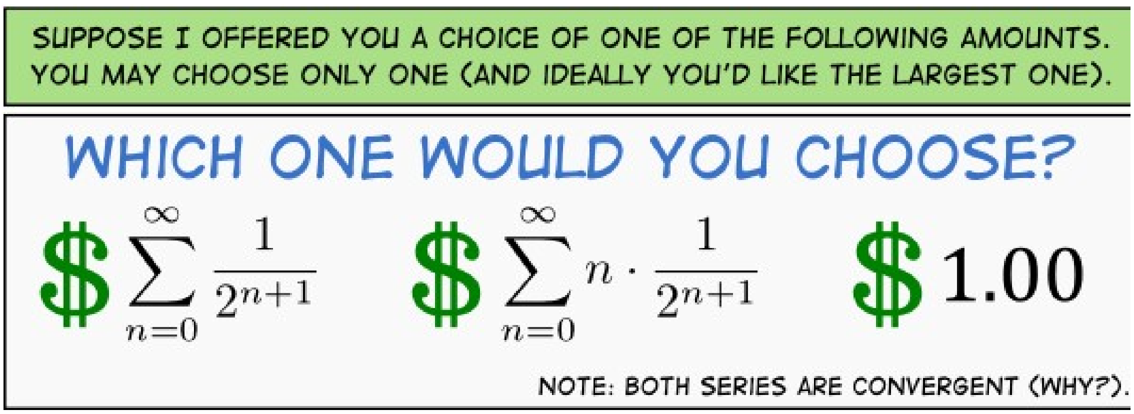
A) $f'(1) = \sum\limits_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$

B) $f'(1) = \sum\limits_{n=1}^{\infty} \frac{1}{2^{n+1}} = 1$

C) $f'(1) = \sum\limits_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$

D) $f'(1) = \sum\limits_{n=1}^{\infty} \frac{n}{2^{n+1}} = \frac{1}{3}$

E) $f'(1) = \sum\limits_{n=1}^{\infty} \frac{n+1}{2^{n+1}} = \frac{3}{2}$



<u>Solution</u> $f'(x) = \sum\limits_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}$, so $f'(1) = \sum\limits_{n=1}^{\infty} \frac{n}{2^{n+1}}$

However $f(x)$ is a geometric series, so we can write $f(x) = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{x}{2}} = \frac{1}{2-x}$

and therefore $f'(x) = \frac{1}{(2-x)^2}$, so $f'(1) = 1$.

Therefore $f'(1) = \sum\limits_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$.

(*In the cartoon, all three are equal.*)

Example $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ is the sum of the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad (\text{for } |x| < 1)$$

Multiplying by $2x$ gives

$$\frac{2x}{1+x^2} = 2(x - x^3 + x^5 - x^7 + x^9 - \ldots) \qquad = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1} \quad (\text{for } |x| < 1)$$

Integrating gives

$$\int \frac{2x}{1+x^2}dx = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1} + C_1 \qquad (\text{for } |x| < 1)$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots + C_1 \quad (\text{for } |x| < 1)$$

$$||$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots + C \qquad (\text{for } |x| < 1)$$

When $x = 0$, this gives $\ln 1 = 0 + C$, so c must be 0.

so we can write

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1} \qquad (\text{for } |x| < 1)$$

Note: we can't justify here, but in fact this equation is also true when $x = 1$. So

$$\mathit{ln}\, 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} \quad \textit{(the alternating harmonic series)}$$

Q2 Write a power series that represents $f(x) = \frac{1}{1+x^6}$ for $|x| < 1$

A) $1 + x^6 + x^{12} + x^{18} + x^{14} + \cdots \qquad = \sum_{n=0}^{\infty} x^{6n}$

B) $1 - x^2 + x^4 - x^6 + x^8 + \cdots \qquad = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

C) $1 + x^2 + x^4 + x^6 + x^8 \cdots \qquad = \sum_{n=0}^{\infty} x^{2n}$

D) $1 - x^6 + x^{12} - x^{18} + x^{24} - \cdots \qquad = \sum_{n=0}^{\infty} (-1)^n x^{6n}$

E) $1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \qquad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

Solution $f(x) = \frac{1}{1+x^6} = \frac{1}{1-(-x^6)} =$ sum of geometric series with $a = 1$, and $r = -x^6$:

$$\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{6n} \qquad \text{where } |r| = |x^6| < 1,$$

that is, where $|x| < 1$

Example Approximate the value of $\int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx$. We could use the Midpoint Rule, or Simpson's Rule, or, here's another method involving infinite series (which is very efficient in this example):

$$\int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx = \int_0^{1} 1 - x^6 + x^{12} - x^{18} + \ldots \, dx = \left(x - \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \ldots\right)\Big|_0^{\frac{1}{4}} =$$

$$= \frac{1}{4} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^{13}}{13} - \frac{(\frac{1}{4})^{19}}{19} + \ldots$$

This is a convergent alternating series, and if we approximate the sum using just the first three terms we get

$$\int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx \qquad \approx \frac{1}{4} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^{13}}{13} \qquad \approx 0.249991281838207$$

and since this is an alternating series, we know that

$$\left| \int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx - \left(\frac{1}{4} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^{13}}{13}\right) \right| = |\text{error}| \leq \frac{(\frac{1}{4})^{19}}{19}$$

so that $\left(\frac{1}{4} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^{13}}{13}\right) - \frac{(\frac{1}{4})^{19}}{19} \leq \int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx \leq \left(\frac{1}{4} - \frac{(\frac{1}{4})^7}{7} + \frac{(\frac{1}{4})^{13}}{13}\right) + \frac{(\frac{1}{4})^{19}}{19}$

Since $\frac{(\frac{1}{4})^{19}}{19} \approx 1.914725687943007 \times 10^{-13}$ we get

$$\mathbf{0.249991281838}016 \leq \int_0^{\frac{1}{4}} \frac{1}{1+x^6}\, dx \leq \mathbf{0.249991281838}399$$

This is very good accuracy for very little numerical computation – because the terms in this particular alternating series approach 0 so quickly!