We have seen that we can sometimes write a function as a power series centered at some point a. So far, our ability to do this has been limited to cases where we could recognize the function as the sum of a geometric series. For example

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ ... \ = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad\qquad |x| < 1$$

In general, we can try to find a power series representation for a function $f(x)$ centered as a whenever it's possible to compute infinitely many derivatives for f at the point a: $f'(a),\ f''(a),\ f'''(a),\ ...,\ f^{(n)}(a),\ ...$ When we try, though, we need to be careful about the conclusions to draw.

To begin with, let's suppose that it is possible to find a power series centered at a that represents $f(x)$ in some interval centered at a. If it's possible, what would that power series have to look like?

IF IT's POSSIBLE TO WRITE

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \ ...$$

THEN (inside the interval of convergence):

$$\begin{aligned}
f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \ ...\\
f''(x) &= 2c_2 + 3(2)c_3(x-a) + (4)(3)c_4(x-a)^2 + (5)(4)c_5(x-a)^3 + ...\\
f'''(x) &= 3(2)c_3 + (4)(3)(2)c_4(x-a) + (5)(4)(3)c_5(x-a)^2 + ...\\
f''''(x) &= (4)(3)(2)c_4 + (5)(4)(3)(2)c_5(x-a) + ...\\
&\vdots
\end{aligned}$$

When $x = a$, all the terms containing an $(x-a)$ factor become 0, so

$$\begin{aligned}
f(a) &= c_0 & & c_0 = f(a)\\
f'(a) &= c_1 & & c_1 = f'(a) = \tfrac{f'(a)}{1!}\\
f''(a) &= 2c_2 & \underline{\text{so}} \qquad & c_2 = f''(a)/2 = \tfrac{f''(a)}{2!}\\
f'''(a) &= 3 \cdot 2c_3 & & c_3 = f'''(a)/(3 \cdot 2) = \tfrac{f'''(a)}{3!}\\
f''''(a) &= 4 \cdot 3 \cdot 2c & & c_4 = f''''(a)/(4 \cdot 3 \cdot 2) = \tfrac{f''''(a)}{4!}\\
&\vdots & & \vdots\\
f^{(n)}(a) &= n(n-1)...(2)c_n & & c_n = \tfrac{f^{(n)}(a)}{n!}
\end{aligned}$$

(for neatness, we can also write $c_0 = \frac{f^{(0)}(a)}{0!}$ if we agree to use $f^{(0)}(x) =$ "the 0^{th}derivative" for $f(x)$ and agree to the convention that $0! = 1$)

So we get that **IF IT's POSSIBLE TO WRITE**

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \ ...$$

then it <u>must be</u> that the series has each $c_n = \frac{f^{(n)}(a)}{n!}$, so the series <u>must be</u>

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ ... \ + \ \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

$= \sum_{n=-}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$: this is called the <u>Taylor series for $f$centered at a.</u>

It is the <u>only candidate</u> for a power series, centered at a, that represents f.

Whether it "works" and for what x's needs to be determined.

Q1: For $f(x) = \frac{1}{1+x^2}$, what is the value of the tenth derivative evaluated at 0 :
$f^{(10)}(0) =$

A) 0 B) $-\frac{1}{10!}$ **C) $-\mathbf{10!}$** D) $\frac{1}{10!}$ E) 1

Solution we already know (from what we know about geometric series) that there is a power series centered at $a = 0$, that represents $f(x)$ for $|x| < 1$:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad |x| < 1$$

Since the Taylor series for $f(x)$, with $a = 0$, is the only candidate for a power series, centered at $a = 0$, that represents f, this series must actually be the Taylor series !!.

So the coefficients in this series must be the Taylor series coefficients $c_n = \frac{f^{(n)}(0)}{n!}$.

If you check you'll find, just for example, that $c_1 = \frac{f'(0)}{1!} = 0$, which is why there is no $c_1 x$ term in the series: x has coefficient $c_1 = 0$.

The coefficient of the x^{10} term in the series is $c_{10} = -1 = \frac{f^{(10)}(0)}{10!}$, so $f^{(10)}(0) = -10!$

Some examples of Taylor series (all of these with $a = 0$) and some things that can happen:

1) For $f(x) = \frac{1}{1+x^2}$

Taylor series at $a = 0$: $1 + x^2 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

Because this is a geometric series, we already know that it converges when $|r| = |-x^2| < 1$, that is, when $|x| < 1$. And, from what we already know about geometric series, that

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

In this example, the Taylor series for the function $f(x)$ converges for some x (not all!) and, where the series converges, its sum is the function $f(x) = \frac{1}{1+x^2}$

2) Very simple example: $f(x) = x^2$. What is the Taylor series centered at $a = 0$?

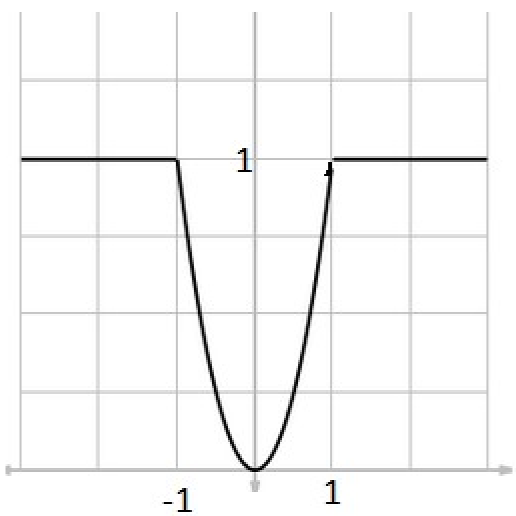
$f(x) = x^2,\ f'(x) = 2x,\ f''(x) = 2,$ and $f^{(n)}(x) = 0$ when $n > 2$.

so $\quad c_0 = \frac{f(0)}{0!} = \frac{0}{1} = 0,\ c_1 = \frac{f'(0)}{1!} = \frac{0}{1!} = 0,\ \ c_2 = \frac{f''(0)}{2!} = \frac{2}{2!} = 1,$ and, for $n > 2$,
$c_n = \frac{f^{(n)}(0)}{n!} = \frac{0}{n!} = 0$. That is, all $c_n = 0$ except for $c_2 = 1$

The Taylor series is $\sum_{n=0}^{\infty} c_n x^n = 0 + 0x + 1x^2 + 0x^3 + 0x^4 + \ldots. \quad = x^2$
Since there's only one nonzero term in the series, it obviously converges for all x.

This example illustrates the "happiest case possible": the Taylor series for $f(x)$ converges for all x and, where the series converges, its sum is the original function $f(x)$.

3) A oddball function. Let $g(x) = \begin{cases} x^2 & \text{when } -1 \le x \le 1 \\ 1 & \text{when } |x| \ge 1 \end{cases}$



Calculating derivatives at $a = 0$ depends only on the values of $f(x)$ near 0 – so the derivatives evaluated at 0 are exactly the same as for the function x^2. Therefore the coefficients in the Taylor series at $a = 0$ are exactly the same as for the function x^2. The Taylor series at 0 for this function $f(x)$ is the same as for $f(x) = x^2$:

The Taylor series is $\sum_{n=0}^{\infty} c_n x^n = 0 + 0x + 1x^2 + 0x^3 + 0x^4 + \ldots. \quad = x^2$
In this example, The Taylor series for $g(x)$ converges for all x, but its sum $(= x^2)$ equals the function $g(x)$ only for $-1 \le x \le 1$.

The preceding examples illustrate that when we write the Taylor series for a function it might converge for some x's or all x's; and even where it does converge, its sum might not be the function that you used to generate the Taylor series ! We need to discuss (next lecture) this further.

For now, here are a couple more examples of Taylor series (at $a = 0$), without comment about whether the function sums up to the original function $f(x)$.

Example What is the Taylor series at $a = 0$ for the function $f(x) = e^x$?

Q2: In the Taylor series for $f(x) = e^x$ centered at $a = 0$

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \; .....$$

What is the coefficient c_3?

A) 1 B) -1 C) $\frac{1}{3}$ D) $\frac{1}{4}$ **E) $\frac{1}{6}$**

Since $f^{(3)}(x) = e^x$, we get that $c_3 = \frac{f^{(3)}(0)}{3!} = \frac{e^0}{3!} = \frac{1}{3!} = \frac{1}{6}$

In general, $f^{(n)}(x) = e^x$ for every n, we get a formula for all the c_n's very easily:

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$$

The full Taylor series at $a = 0$ is

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \; ... \; = \frac{1}{01} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... + \frac{1}{n!}x^n + ...$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \; ... + \frac{x^n}{n!} - ... \; = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Notice that by the Ratio Test, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0$ for every x. The Taylor series converges for all x (but whether its sum is e^x is still uncertain).

Example What is the Taylor series at $a = 0$ for the function $f(x) = \sin x$?

Here, the derivatives repeat in "blocks" of size 4:

$f(x) = \sin x$	$f^{(4)}(x) = \sin x$	$f^{(8)}(x) = \sin x$
$f'(x) = \cos x$	$f^{(5)}(x) = \sin x$	$f^{(9)}(x) = \sin x$ etc.
$f''(x) = -\sin x$	$f^{(6)}(x) = \sin x$	$f^{(10)}(x) = \sin x$
$f'''(x) = -\cos x$	$f^{(7)}(x) = \sin x$	$f^{(11)}(x) = \sin x$

So the derivatives evaluated at 0 also repeat in blocks:

$$\begin{array}{lll} f(0)=0 & f^{(4)}(0)=0 & f^{(8)}(0)=0 \\ f'(0)=1 & f^{(5)}(0)=1 & f^{(9)}(0)=1 \quad \text{etc.} \\ f''(0)=0 & f^{(6)}(0)=0 & f^{(10)}(0)=0 \\ f'''(0)=-1 & f^{(7)}(0)=-1 & f^{(11)}(0)=-1 \end{array}$$

so the numerators for the c_n's repeat in blocks, giving

$$\begin{array}{llll} c_0=\frac{0}{0!} & c_4=\frac{0}{4!} & c_8=\frac{0}{8!} & \\ c_1=\frac{1}{1!} & c_5=\frac{1}{5!} & c_9=\frac{1}{9!} & \text{etc.} \\ c_2=\frac{0}{2!} & c_6=\frac{0}{6!} & c_{10}=\frac{0}{10!} & \\ c_3=\frac{-1}{3!} & c_7=\frac{-1}{7!} & & c_{11}=\frac{-1}{11!} \end{array}$$

Therefore all the even powers x^0, x^2, x^4, x^6, ... have coefficient 0 and these terms "drop out" of the Taylor series. The Taylor series for sin x at $a=0$ is

$$c_1x+c_3x^3+c_5x^5+\ldots = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\ \ldots\ =\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

The Ratio Test: $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim_{n\to\infty}|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}|=\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)}|x^2|=0$ for any value of x.

The Taylor series converges for all x (but whether its sum is e^x is still uncertain.