Q1: For what x's does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

A) $-1 < x < 1$ B) $-1 \le x \le 1$ C) $-e < x < e$

D) $-e \le e \le e$ **E) all $\boldsymbol{x}$**

Solution Use the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0 \text{ for any value of } x.$$

Since the limit is < 1 for every x, the ratio test says that the series converges (absolutely) for all x

Note: for the last lecture, this is the Taylor series, centered at $a = 0$ for the function $f(x) = e^x$. But even though the series converges for all x, it might not be true that the sum of the series is e^x. (See last lecture where for a very simple example of where the Taylor series (centered at 0) for a function converges for all x, but not to the original function.)

Q2: What is $\lim_{n\to\infty} \frac{1000^n}{n!}$?

A) 0 B) 1 C) 1000 D) ∞

What do you conclude about $\lim_{n\to\infty} \frac{x^n}{n!}$?

Solution You can reason this out directly, but the simplest solution is to note that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1000^n}{n!}$ converges (see Q1), and therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1000^n}{n!} = 0.$

(*Recall the Test for Divergence\: if $\lim_{n\to\infty} a_n \neq 0,$ the series would have to diverge!*)

The same argument for any value for x, works because $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

The “moral” to take away for use late in the lecture is that the limit is 0 because

“factorial goes to ∞ faster than the powers of x (for any x)”

Assuming that its possible to find the n^{th} derivative $f^{(n)}(a)$ for every $n = 0, 1, 2, ...$ then it makes sense to write down a power series called the Taylor series of f centered at a :

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)^1}(x-a)^{n+1} + \cdots$$

Usually applying the Ratio Test will tell you the radius of convergence of the series. But, whatever that turns out to be, the question is still open: inside the interval of convergence, is the sum of the Taylor series $= f(x)$ or not?

To answer this, we break the series into "two parts"

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)^1}(x-a)^{n+1} + \cdots$$

partial sum of terms up to the $(x-a)^n$ term + remainder

$= T_n(x)$ + $= R_n(x)$

Just as for any infinite series,

sum of the Taylor series $= f(x)$ is equivalent to

$\lim\limits_{n\to\infty}$ (partial sums) $= \lim\limits_{n\to\infty} T_n(x) = f(x)$ which is equivalent to

$\lim\limits_{n\to 4} f(x) - T_n(x) = 0$ which is equivalent to

$\lim\limits_{n\to\infty} R_n(x) = 0$

So to decide whether $f(x) =$ sum of its Taylor series for any particular x, we need to check whether

$$\lim_{n\to\infty} R_n(x) = 0.$$

To do that, we need something that tells us the "size" of $R_n(x)$. This information comes form the following "Taylor's Inequality."

Taylor's Inequality — If $|f^{(n+1)}(x)| \leq M$ for all x in $[a-d,\ a+d|$,
that is, whenever
$|x-a| \leq d$,

then $|R_n(x)| \leq \frac{M\,|x-a|^{n+1}}{(n+1)!}$

Example Let's consider $f(x) = e^x$ and its Taylor series at $a = 0$:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The first few Taylor polynomials (partial sums) are

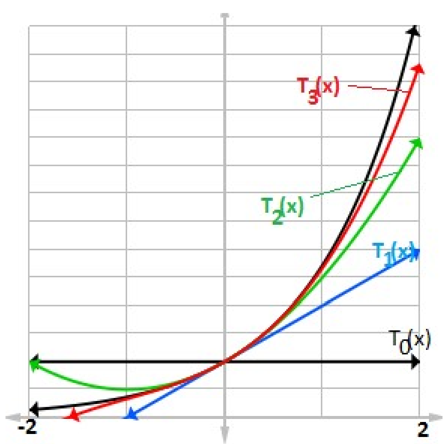
$$T_0(x) = 1$$
$$T_1(x) = 1 + x$$
$$T_2(x) = 1 + x + \frac{x_2}{2!}$$
$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

etc.

$\downarrow$ (as $n \to \infty$)

??? (e^x ???)

Consider the graphs just on the interval $[-2, 2]$ (*centered at* 0):



Graph of e^x and Taylor polynomials T_0, T_1, T_2, T_3 at a = 0

Pick any x in $[-2,2|$: for that x it appears that

$$T_0(x),\ T_1(x),\ T_2(x),\ T_3(x) \ \rightarrow\ e^x.$$

Notice also that when x close to $a = 0$, then even $T_1(x)$ is very close to e^x; when x is farther from 0 *(say, $x = 1.9$), we need a bigger n for $T_n(x)$ to be very close to e^x. It looks like the Taylor polynomials at x are* <u>*better*</u> *approximations to e^x when x is near the center of the Taylor series, that is, near $a = 0$. In fact, every Taylor polynomial is* <u>*exactly*</u> $= e^x$ <u>*at 0*</u>*:* $1 = e^0 = T_0(0) = T_1(0) = ... = T_n(0)$.

Now we'll settle matters using Taylor's Inequality: pick <u>any</u> $d > 0$ and consider the interval $[-d, d]$ (*centered at* 0).
Since, for <u>every</u> n, $f^{(n+1)}(x) = e^x$, and e^x is an increasing function, the largest value of $f^{(n+1)}(x)$ in the interval $[-d, d]$ occurs when $x = d$, that is, $|f^{(n+1)}(x)| \leq e^d$ for every x in $[-d, d]$.

So, by Taylor's Inequality,

$$0 \leq |R_n(x)| \leq \frac{e^d|x-0|^{n=1}}{(n+1)!} = e^d\ \frac{|x|^{n+1}}{(n+1)!} \text{ for any } x \text{ in } [-d, d]$$

e^d is just a constant, so as $n \to \infty$, $e^d\ \frac{|x|^{n+1}}{(n+1)!} \to \mathrm{e}^d(0) = 0$

By the Squeeze theorem, $\lim\limits_{n\to\infty} |R_n(x)| = 0$, so $\lim\limits_{n\to\infty} R_n(x) = 0$.
Therefore, as remarked above,

$$e^x = \text{the sum of its Taylor series for any } x \text{ in } [-d, d] \qquad (*)$$

But d was any positive number we chose at the beginning : d is arbitrary.
So (*) is true for any x i $[-1, 1]$, but also for any x in $[-1000, 1000]$ or ...

This shows that

$$\boxed{\begin{aligned} e^x &= \text{the sum of its Taylor series} \\ &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \tfrac{x^n}{n!} \text{ for any } x \end{aligned}}$$

Q3: The Taylor series for $\cos x$ at $a = 0$ is $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$
What is c_4 ?

A) $\frac{1}{2}/4!$ **B) $1/4!$** C) 0 D) $-1/4!$ E) $-\frac{1}{2}/4!$

Solution $c_n = \frac{f^{(n)}(0)}{n!}$, so:

$f(x) = \cos x$	$f(0) = \cos 0$	$= 1$	$c_0 = 1/0! = 1$
$f'(x) = -\sin x$	$f'(0) = -\sin 0$	$= 0$	$c_1 = 0/1! = 0$
$f''(x) = -\cos x$	$f''(0) = -\cos 0$	$= -1$	$c_2 = -1/2!$
$f'''(x) = \sin x$	$f'''(0) = \sin 0$	$= 0$	$c_3 = 0/3!$

$f^{(4)}(x) = \cos x$ (*repeating in groups of* 4)	$f^{(4)}(0) = \cos 0$	$= 1$	$c_4 = 1/4!$

In general, the numerators of the coefficients repeated in groups of 4:

$$1, 0, -1, 0, 1, 0, -1, 0, 1,$$

so the Taylor series, centered at $a = 0$ for $\cos x$ is

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots$$

$$= 1 + 0x - \frac{x^2}{2!} + 0x^3 + \frac{x^4}{4!} + 0x^5 - \frac{x^6}{6!} + \cdots =$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

In any interval $[-d, d]$, $|f^{n+1)}(x)| \leq 1$ (because $f^{(n+1)}(x)$ is one of $\pm \sin x$ or $\pm \cos x$)and therefore, using Taylor's Inequality, we get

$$0 \leq |R_n(x)| \leq \left| \frac{1 \cdot |x-0|^{n+1}}{(n+1)!} \right| = \left| \frac{x^{n+1}}{n+1} \right|$$

Since $\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \right| = 0$ (*see comment after iClicker Q2*), the Squeeze Theorem shows that $\lim_{n\to\infty} |R_n(x)| = 0$, so $\lim_{n\to\infty} R_n(x) = 0$. Therefore $\cos x =$ sum of its Taylor series at 0 for any x in $[-d.d]$. Since d is an arbitrary positive number, this shows that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x.$$

(A similar argument shows that $\sin x$ is the sum of its Taylor series at 0 for all x.)

Three Very Important Taylor Series (at $a = 0$)

$f(x) = e^x$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \underbrace{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}_{T_n(x)} + \underbrace{\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots}_{R_n(x)}$$

Ratio Test shows that this series converges for <u>all x</u>, but is the sum e^x ?

We used Taylor's Inequality to show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \underline{\text{for all } x}$$

$f(x) = \cos x$

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + (-1)^n \frac{x^{2n}}{(2n)!}}_{T_{2n}(x)} + \underbrace{(-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} + \cdots}_{R_{2n}(x)}$$

Ratio Test shows that this series converges for <u>all x</u>, but is the sum $\cos x$?
We used Taylor's Inequality to show that

$$\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \quad \underline{\text{for all } x}$$

$f(x) = \sin x$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \underbrace{x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{T_{2n+1}(x)} + \underbrace{(-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} + \cdots}_{R_{2n+1}(x)}$$

Ratio Test shows that this series converges for <u>all x</u>, but is the sum $\sin x$?

We could use Taylor's Inequality (just as we did for $\cos x$) to show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \quad \underline{\text{for all } x}$$

Example

$$\sin x \quad = x - \frac{x^3}{3\,!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Since this is true for all x we can substitute x^2 for x in the series to get

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots$$

so $\quad \int_0^1 \sin(x^2)\,dx \quad = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots\right) dx$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \frac{x^{19}}{19\cdot 9!} - \cdots\right)\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} - \cdots$$ (*exact sum of series* $=$ *exact value of integral*)

This is an alternating series, so we can approximate the integral

$\int_0^1 \sin(x^2)\,dx \;\approx\; \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!}$ and we can say about the error that

$$\left|\,\int_0^1 \sin(x^2)\,dx - \left(\frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!}\right)\right| \le \frac{1}{19\cdot 9!} \approx 1.5 \times 10^{-7}$$