Some Taylor Series (at $a = 0$)

1) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x| < 1$

2) $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$

3) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for all x

4) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x

5) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$ for all x

There series are "pretty" but also are useful as a starting point to obtain many others by integration, differentiation, multiplication, or substitution (which might affect the interval of convergence). For example,

Substitute $3x$ for x in 1) to get

$$\frac{1}{1+9x^2} = 1 - 9x^2 + 81x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}$$

Since we substituted $3x$ for x and $|x| < 1$ was originally required, convergence is now for $|3x| < 1$, that is, for $|x| < \frac{1}{3}$for $|3x| < 1$, that

is, for $|x| < \frac{1}{3}$

Integrate 1) to get

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

(*what happened to the constant of integration, C ?*)

Substitute x^2 for x in 5), then multiply the series by x to get

$$x \cos x^2 = x\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!} \quad \text{for all } x$$

Taylor series for $\cos x$ centered at 0

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad\qquad \text{for all } x$$

$$T_0(x) = 1$$

$$T_1(x) = 1 + 0x = 1$$

$$T_2(x) = 1 - \frac{x^2}{2}$$

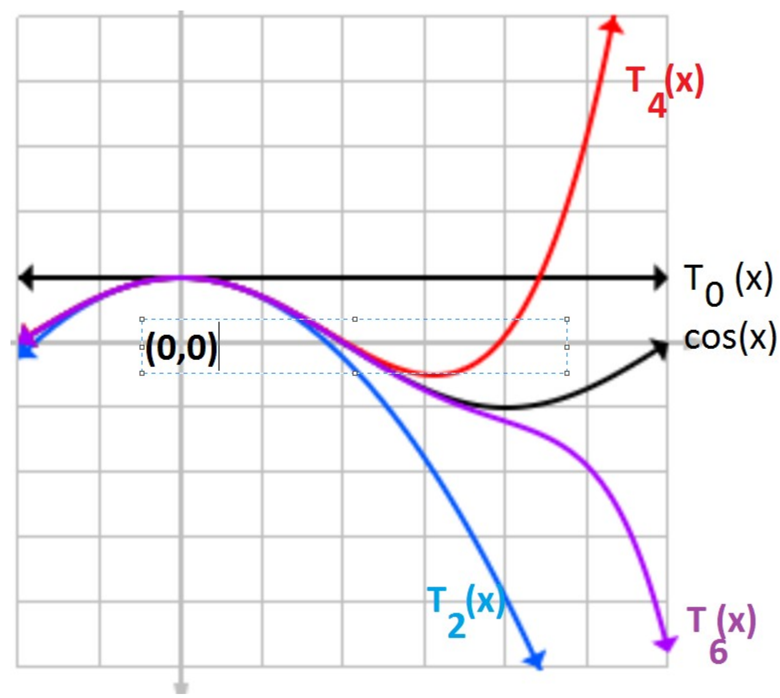
$$T_3(x) = 1 - \frac{x^2}{2} + 0x^3$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\vdots$$

$$\downarrow \qquad\qquad \text{for all } x$$

$$\cos x$$



Notice that for a fixed x value: $T_n(x) \to \cos x$ as $n \to \infty$

and that for a fixed n value, the approximation

$\cos x \approx T_n(x)$ is better (smaller error) when x is near $a = 0$ and worse (bigger error) when x is further from 0.

If we want to approximate $\cos x$ for some specific x, then we'll get a good approximation with less work (that is, using a smaller n) if we use the Taylor series centered as a point a that's close to x.

For example, suppose we want to approximate $\cos 46° =$ (in radian measure) $\cos \frac{46\pi}{180}$. To do this efficiently we should write a Taylor series for $\cos x$ centered at a point a close to $\frac{46\pi}{180}$: a good choice is $a = \frac{\pi}{4} = \frac{45\pi}{180}$.

Q1: In the Taylor series for $\cos x$ centered at $a = \frac{\pi}{4}$:

$$c_0 + c_1(x - \tfrac{\pi}{4}) + c_2(x - \tfrac{\pi}{4})^2 + c_3(x - \tfrac{\pi}{4})^3 + c_4(x - \tfrac{\pi}{4})^4 + \cdots$$

What is c_4?

A) $-\frac{1}{4!}$ B) $\frac{1}{4!}$ C) $-\frac{1}{4!\sqrt{2}}$ D) $\frac{1}{4!\sqrt{2}}$ E) $\frac{1}{5!\sqrt{2}}$

Solution The Taylor series for $\cos x$ centered at $\frac{\pi}{4}$ is

$$c_0 + c_1(x - \tfrac{\pi}{4}) + c_2(x - \tfrac{\pi}{4})^2 + c_3(x - \tfrac{\pi}{4})^3 + c_4(x - \tfrac{\pi}{4})^4 + \cdots$$

The coefficient $c_n = \frac{f^{(n)}(\frac{\pi}{4})}{n!}$.

$$\begin{array}{llll} f(x) = \cos x & f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} & c_0 = \frac{1}{\sqrt{2}} \\ f'(x) = -\sin x & f'(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} & c_1 = -\frac{1}{\sqrt{2}} \\ f''(x) = -\cos x & f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}} & c_2 = -\frac{1}{2!\sqrt{2}} \\ f'''(x) = \sin x & f'''(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} & c_3 = \frac{1}{3!\sqrt{2}} \end{array}$$

(*begin repeating in groups of 4*)

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}} \qquad c_4 = \tfrac{1}{4!\sqrt{2}}$$

so

$$\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \tfrac{\pi}{4}) - \frac{1}{2!\sqrt{2}}(x - \tfrac{\pi}{4})^2 + \frac{1}{3!\sqrt{2}}(x - \tfrac{\pi}{4})^3 + \frac{1}{4!\sqrt{2}}(x - \tfrac{\pi}{4})^4 - \cdots$$

(*the fact that the sum of the series* $=$ *cos* x *for every* x *can be shown, using Taylor's Inequality, in the same way as was done in the preceding lecture were we had* $a = 0$.)

Here are the first few Taylor polynomials for $\cos x$ centered at $a = \frac{\pi}{4}$:

$$T_0(x) = \frac{1}{\sqrt{2}}$$

$$T_1(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \tfrac{\pi}{4})$$

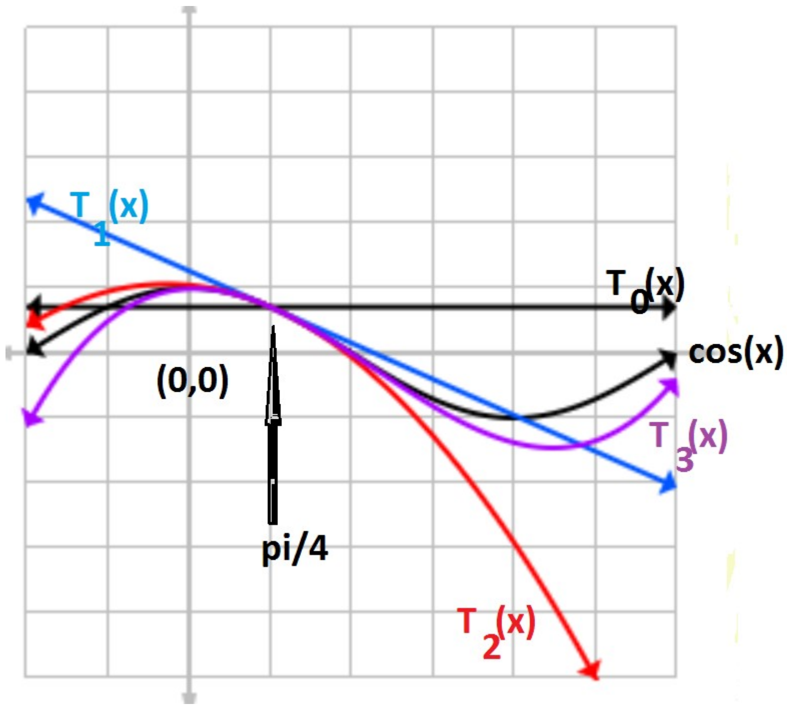
$$T_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \tfrac{\pi}{4}) - \frac{1}{2!\sqrt{2}}(x - \tfrac{\pi}{4})^2$$

$$T_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \tfrac{\pi}{4}) - \frac{1}{2!\sqrt{2}}(x - \tfrac{\pi}{4})^2 + \frac{1}{3!\sqrt{2}}(x - \tfrac{\pi}{4})^3$$

$$\vdots$$

$$\downarrow \qquad \textit{(for all } x\textit{)}$$

$$\cos x$$



Notice that for a fixed x value: $T_n(x) \to \cos x$ as $n \to \infty$
and that

for a fixed n value, the approximation

$\cos x \approx T_n(x)$ is better (smaller error) when x is near $a = \frac{\pi}{4}$
and worse (bigger error) when x is further from $\frac{\pi}{4}$

Example Approximate cos $46°$. For doing calculus with trig functions, we always need to use radian measure, so we convert cos $46° = \cos\frac{46\pi}{180}$. We use a Taylor series with $a = \frac{\pi}{4} = \frac{45\pi}{180}$, a point where its easy to find the coefficients for the Taylor series and also close to $\frac{46\pi}{180}$: $x - a = \frac{46\pi}{180} - \frac{45\pi}{180} = \frac{\pi}{180}$

$$\cos\frac{46\pi}{180} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right) - \frac{1}{2!\sqrt{2}}\left(\frac{\pi}{180}\right)^2 + \frac{1}{3!\sqrt{2}}\left(\frac{\pi}{180}\right)^3 + \frac{1}{4!\sqrt{2}}\left(\frac{\pi}{180}\right)^4 - \cdots$$

This is an alternating series and we can approximate, say,

$\cos\frac{46\pi}{180} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right) - \frac{1}{2!\sqrt{2}}\left(\frac{\pi}{180}\right)^2$ and the error is

$$\left|\cos\frac{46\pi}{180} - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right) - \frac{1}{2!\sqrt{2}}\left(\frac{\pi}{180}\right)^2\right)\right| < \frac{1}{3!\sqrt{2}}\left(\frac{\pi}{180}\right)^3 \approx 6.3 \times 10^{-7}$$

Because $\frac{46\pi}{180}$ is so close to $a = \frac{\pi}{4} = \frac{45\pi}{180}$, we get a very good approximation (small error) just using the Taylor polynomial $T_2(x)$.

In fact, $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{180}\right) - \frac{1}{2!\sqrt{2}}\left(\frac{\pi}{180}\right)^2 = 0.694658$ (*rounded to 6 decimal places*)

This is exactly the same value a calculator gives for cos $46°$ (calculator set for degree measure, and answer rounded to 6 decimal places).

Example Sometimes a Taylor series calculation can replace L'Hospital's Rule.

$$\lim_{x\to 0}\frac{e^x - e^{-e}}{x} = \lim_{x\to 0}\frac{1}{x}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \left(\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right)\right)$$

$$= \lim_{x\to 0}\frac{1}{x}\left(2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \dots\right) = \lim_{x\to 0}\left(2 + 2\frac{x^2}{3!} + 2\frac{x^4}{5!} + \dots\right) = 2$$

(*This is the same answer as you'd get with L'Hospital's Rule*)

Example

Q2: What is $\lim\limits_{x \to 0} \frac{\sin x^2 - x^2}{x^2}$ (*use power series, not L'Hospital's Rule*)

A) 0 B) 1 C) -1 D) $\frac{1}{2}$ E) d.n.e.

Solution Write sin x^2 as a Taylor series (substitute x^2 for x in the series for x^2) and then simplify:

$$\lim_{x \to 0} \frac{\sin x^2 - x^2}{x^2} = \lim_{x \to 0} \frac{1}{x^2}\left(\left(x^2 - \frac{x^6}{5!} + \frac{x^{10}}{5!} - \ldots\right) - x^2 \right)$$

$$= \lim_{x \to 0} \frac{1}{x^2}\left(\left(-\frac{x^6}{5!} + \frac{x^{10}}{5!} - \ldots\right)\right.$$

$$= \lim_{x \to 0} \left(-\frac{x^4}{5!} + \frac{x^8}{5!} - \ldots\right) = 0$$

(*This is the same answer as you'd get with L'Hospital's Rule*)

Final example (meant for fun)

The series that we discussed above also work for complex numbers $z = a + bi$ (where $i^2 = -1$).

For example,

3) $e^z = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{x^{zn}}{n!} + \cdots$ for all complex numbers $z = a + bi$

Let $z = ix$ (where x is a real number) and rearrange:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots\cdots$$

Since $i^2 = -1,\ i^4 = 1,\ 1^6 = -1,$ etc

$$= \left(1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \cdots\right) + \left(ix + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \right)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(ix + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \right)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x + \frac{i^2x^3}{3!} + \frac{i^4x^5}{5!} + \right)$$

$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ )$, so

$e^{ix} \ = \cos x \ + i \sin x$ a nice connection between
the eponential and trig functions

Letting $x = \pi,$ we get

$e^{i\pi} = \cos \pi + i \sin \pi$

$e^{i\pi} = 1 + i(0)$

$e^{i\pi} = 1$

$e^{i\pi} - 1 = 0$ a formula that connects the 5 most important constants in mathematics!

We conclude with the final speech of Robin Goodfellow (Puck) in Shakespeare's *A Midsummer Night's Dream*

ROBIN

If we shadows have offended,
Think but this, and all is mended—
That you have but slumbered here
While these visions did appear.
And this weak and idle theme,
No more yielding but a dream,
Gentles, do not reprehend.
If you pardon, we will mend.
And, as I am an honest Puck,
If we have unearnèd luck
Now to 'scape the serpent's tongue,
We will make amends ere long.
Else the Puck a liar call.
So good night unto you all.
Give me your hands if we be friends,
And Robin shall restore amends.

Good luck!