Theorem Suppose A is a real 2×2 matrix with a complex eigenvalue $a - bi$ and a corresponding eigenvector $\boldsymbol{v}$. Then $A = PCP^{-1}$

where $P = [\,\text{Re}\,\boldsymbol{v}\;\; \text{Im}\,\boldsymbol{v}\,]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Interpretation:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ can be written as } r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } r = \sqrt{a^2 + b^2}.$$

Thus C represents a counterclockwise rotation (if θ is chosen > 0) around the origin through the angle θ, <u>followed by</u> a rescaling factor of r.

If we use $\mathcal{B} = \{\text{Re}\,\boldsymbol{v},\; \text{Im}\,\boldsymbol{v}\}$ as a new basis for $\mathbb{R}^2$, then the change of coordinate matrix $P_{\mathcal{B}} = P = [\,\text{Re}\,\boldsymbol{v}\;\; \text{Im}\,\boldsymbol{v}\,]$.

The effect of A, broken into several steps, is then:

$$\boldsymbol{x} \quad \overset{\text{switch to } \mathcal{B}\text{-coordinates}}{\longmapsto} \quad P^{-1}\boldsymbol{x} = [\boldsymbol{x}]_{\mathcal{B}} \quad \overset{\text{rotate and dilate by a factor of } r \text{ in the new coordinates}}{\longmapsto} \quad C[\boldsymbol{x}]_{\mathcal{B}} = CP^{-1}\boldsymbol{x}$$

$$\overset{\text{switch back to standard coordinates}}{\longmapsto} \quad PC[\boldsymbol{x}]_{\mathcal{B}} = PCP^{-1}\boldsymbol{x} = A\boldsymbol{x}$$

1) If $r = 1$, C represents a “pure“ rotation (in the new coordinates)

2) If $r > 1$, then the successive images $\boldsymbol{x_0}$, $\boldsymbol{x_1} = A\boldsymbol{x_0}$, ..., $\boldsymbol{x_{n+1}} = A\boldsymbol{x_n}$, ... move further and further away from the origin (assuming $\boldsymbol{x_0} \neq \boldsymbol{0}$)

3) If $r < 1$, then the successive images $\boldsymbol{x_0}$, $\boldsymbol{x_1} = A\boldsymbol{x_0}$, ..., $\boldsymbol{x_{n+1}} = A\boldsymbol{x_n}$, ... approach the origin.

Example Let $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$. The characteristic equation is

$$\det\begin{bmatrix} \frac{1}{2}-\lambda & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4}-\lambda \end{bmatrix} = (\tfrac{1}{2}-\lambda)(\tfrac{5}{4}-\lambda)+\tfrac{3}{8} = \tfrac{5}{8}-\tfrac{7}{4}\lambda+\lambda^2+\tfrac{3}{8}$$

$= \lambda^2 - \frac{7}{4}\lambda + 1 = 0$, which has the same solutions as $4\lambda^2 - 7\lambda + 4 = 0$

The eigenvalues are $\lambda = \frac{7 \pm \sqrt{49-64}}{8} = \frac{7}{8} \pm \frac{\sqrt{-15}}{8} = \frac{7}{8} \pm \frac{\sqrt{15}}{8}\, i$.

For no particular reason, <u>choose the eigenvalue</u> $\lambda = \frac{7}{8} - \frac{\sqrt{15}}{8}\, i$.

<u>To find the corresponding eigenspace</u>: solve $(A-\lambda I)\boldsymbol{x} = \mathbf{0}$

$$\begin{bmatrix} \frac{1}{2}-(\frac{7}{8}-\frac{\sqrt{15}}{8}\,i) & -\frac{1}{2} & 0 \\ \frac{3}{4} & \frac{5}{4}-(\frac{7}{8}-\frac{\sqrt{15}}{8}\,i) & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{8}+\frac{\sqrt{15}}{8}\,i & -\frac{1}{2} & 0 \\ \frac{3}{4} & \frac{3}{8}+\frac{\sqrt{15}}{8}\,i & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3+\sqrt{15}\,i & -4 & 0 \\ 6 & 3+\sqrt{15}\,i & 0 \end{bmatrix}.$$

<u>A very handy observation</u>: the task can now be simplified because when we set out to solve $(A-\lambda I)\boldsymbol{x} = \mathbf{0}$, we <u>already</u> know that there are nontrivial solutions because we already know that det$(A-\lambda I) = 0$, that is, that λ <u>is</u> an eigenvalue. Since $A-\lambda I$ is not invertible, this means, that its rows (columns) are linearly dependent. <u>In the</u> 2×2 <u>case</u>, that means that <u>one of the rows in the augmented matrix is a multiple of the other</u> and therefore <u>each of the two equations states the same relationship between</u> x_1 <u>and</u> x_2. We can simply use (either) one of the equations to see the relationship and find the eigenspace.

The first equation says

$$(-3+\sqrt{15}\,i)x_1 - 4x_2 = 0, \qquad \text{that is,} \qquad x_2 = \frac{-3+\sqrt{15}\,i}{4}\,x_1.$$

To find an eigenvector, we can just choose $x_1 = 1$ and get $\begin{bmatrix} 1 \\ \frac{-3+\sqrt{15}\,i}{4} \end{bmatrix}$. A neater eigenvector would be 4 times this one: $\boldsymbol{v} = \begin{bmatrix} 4 \\ -3+\sqrt{15}\,i \end{bmatrix}$.

In the notation of the Theorem, we have:

- $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$

- An eigenvalue $\lambda = \frac{7}{8} - \frac{\sqrt{15}}{8}i$ (so $a = \frac{7}{8}$, $b = \frac{\sqrt{15}}{8}$), and

- A corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix} 4 \\ -3 + \sqrt{15}\, i \end{bmatrix} = \begin{bmatrix} 4 + 0 \cdot i \\ -3 + \sqrt{15}\, i \end{bmatrix}$, for

  For this $\boldsymbol{v}$: Re $\boldsymbol{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and Im $\boldsymbol{v} = \begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$.

The main theorem, above (*Theorem 9 in the text*) states that A factors as PCP^{-1}, where

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix} \text{ and}$$

$$P = [\,\text{Re}\,\boldsymbol{v} \ \ \text{Im}\,\boldsymbol{v}\,] = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}, \text{ that is}$$

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix} \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}^{-1}.$$

I checked myself for errors using Matlab: rounded to 4 places, Matlab gives $AP = \begin{bmatrix} 3.5 & -1.9365 \\ -.75 & 4.8412 \end{bmatrix} = PC$

What does this mean geometrically?

Take $\mathcal{B} = \{\mathrm{Re}\,\boldsymbol{v}\ \ \mathrm{Im}\,\boldsymbol{v}\ \} = \{ \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix} \}$ as a new basis.

$$\underset{\boldsymbol{b_1}}{\uparrow} \qquad \underset{\boldsymbol{b_2}}{\uparrow}$$

Write $\mathcal{B}$-coordinates as x', y'. Then P is the change of coordinates matrix from $\mathcal{B}$-coordinates to standard coordinates:

$$P[\boldsymbol{x}]_B = \boldsymbol{x}, \quad \text{that is}$$

$$P\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

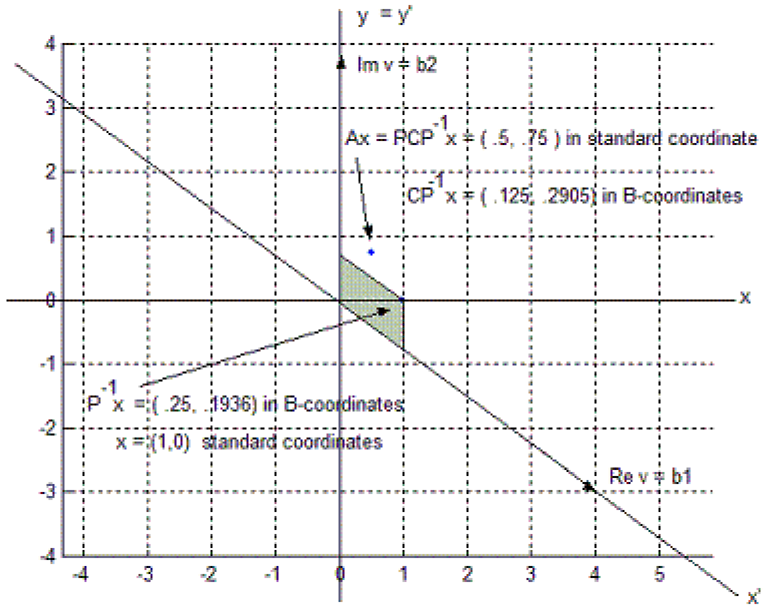
As we noted in the last lecture, we can always rewrite a matrix like C as

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ where } r = \sqrt{a^2 + b^2}.$$

In this example , $r = \sqrt{a^2 + b^2} = \sqrt{\frac{49}{64} + \frac{15}{64}} = \sqrt{\frac{64}{64}} = 1$, so

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 1 \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix}$$

C represents a “pure rotation” (because the rescaling factor $r = 1$). So $\cos\theta = \frac{7}{8}$ and $\sin\theta = \frac{\sqrt{15}}{8}$. From Matlab or a calculator, we can choose $\theta \approx 0.5054$ (radians) $\approx 28.96°$.



So: if we start with $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

P^{-1} converts into $\mathcal{B}$-coordinates: $P^{-1}\boldsymbol{x} =$ the $\mathcal{B}$-coordinate vector of the same point: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} .25 \\ .1936 \end{bmatrix}$

Then C rotates: $CP^{-1}\boldsymbol{x} =$ the location of the point (in $\mathcal{B}$-coordinates) after the rotation $= \begin{bmatrix} .125 \\ .2905 \end{bmatrix}$

Then P converts back to standard coordinates. $A\boldsymbol{x} = PCP^{-1}\boldsymbol{x} =$ the location of the coordinates point (in standard coordinates) after the rotation: $\begin{bmatrix} .5 \\ .75 \end{bmatrix}$

An alternate way of picturing the action of A :   this version plots everything in the standard x-y plane:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P^{-1}\boldsymbol{x} = \begin{bmatrix} .25 \\ .1936 \end{bmatrix}$$

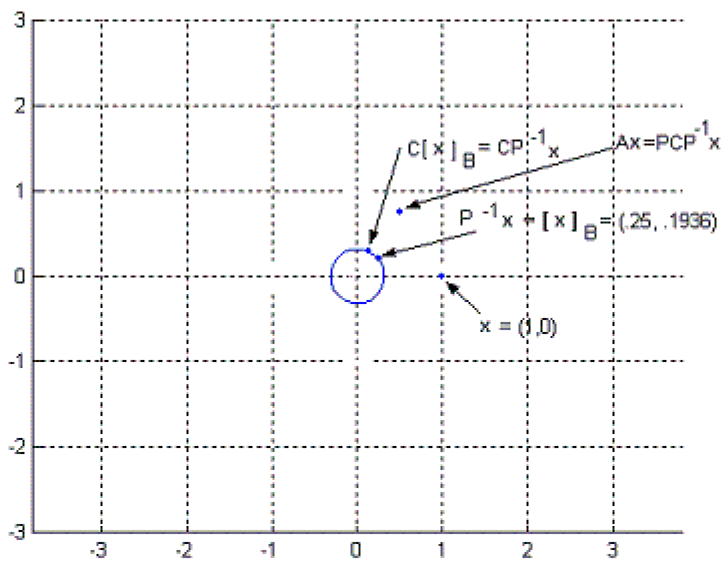
(*this is the $\mathcal{B}$-coordinate vector of* $\boldsymbol{x}$*, but it is plotted below as a point in the* x-y *plane*)

$$CP^{-1}\boldsymbol{x} = \begin{bmatrix} .125 \\ .2905 \end{bmatrix}$$ is the location after rotation by $\theta \approx 28.96°$

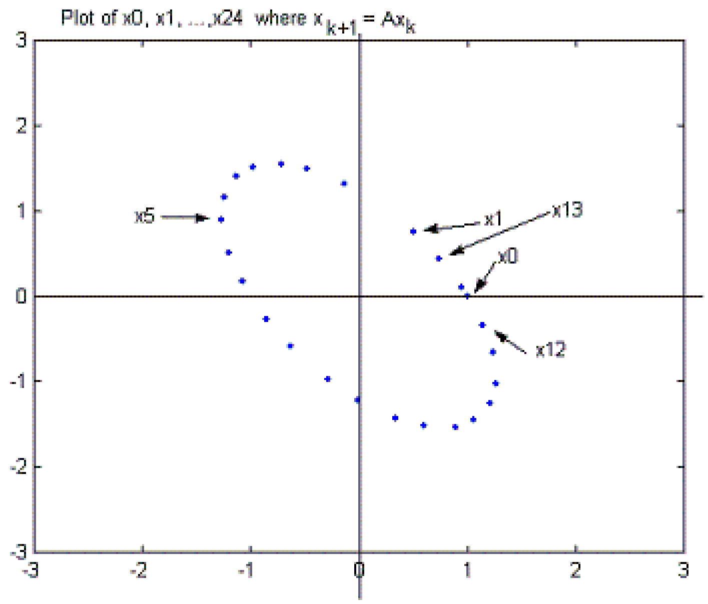
(*this is also a $\mathcal{B}$-coordinate vector, but it's plotted in the* x-y *plane*)

$$A\boldsymbol{x} = PCP^{-1}\boldsymbol{x} = \begin{bmatrix} .5 \\ .75 \end{bmatrix}$$ is the final location of the point

(*in standard coordinates and plotted in the* x-y *plane)*



The figure below shows the successive iterates if we start with $\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and plot $\boldsymbol{x}_0$, $\boldsymbol{x}_1 = A\boldsymbol{x_0}$, $\boldsymbol{x_2} = A\boldsymbol{x_1}$, ..., $\boldsymbol{x_{24}} = A\boldsymbol{x_{23}}$.



The images run along an elliptical "orbit" $-$ even though C is a "pure" rotation $(r = 1)$, the new coordinate basis vectors $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \sqrt{15} \end{bmatrix}$ are not perpendicular and have different lengths, so the orbit isn't a circle.

To briefly illustrate the case where $r > 1$, let $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}$

$= 4*$(matrix A in the preceding example)

Multiplying a square matrix by 4 multiplies the eigenvalues by 4 but doesn't change the eigenvectors (*why? – check this from the definitions of eigenvalue and eigenvector*)

So $A = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}$ has a complex eigenvalue $\lambda = \frac{7}{2} - \frac{\sqrt{15}}{2}\, i$ with a corresponding

eigenvector $\begin{bmatrix} 4 \\ -3 + \sqrt{15}\, i \end{bmatrix}$. According to the Theorem, we can write

$$\begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix} \begin{bmatrix} \frac{7}{2} & -\frac{\sqrt{15}}{2} \\ \frac{\sqrt{15}}{2} & \frac{7}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -3 & \sqrt{15} \end{bmatrix}^{-1}.$$

I checked myself for errors using Matlab: rounded to 4 places, Matlab gives

$$AP = \begin{bmatrix} 14 & -7.7460 \\ -3 & 19.3649 \end{bmatrix} = PC$$

Then write $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = 4 \begin{bmatrix} \frac{7}{8} & -\frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix}$

$$= 4 \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\cos\theta = \frac{7}{8}$, $\sin\theta = \frac{\sqrt{15}}{8}$. Again, we can choose $\theta \approx 0.5054$ or $\theta \approx 28.96°$. The matrix C rotates the new coordinate vector and then rescales it by a factor of 4.

The following figure illustrates the first few iterations, starting with $\boldsymbol{x_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: