Example: Orthogonal Decomposition Theorem

Let $W = \text{Span}\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\} \subseteq \mathbb{R}^4$. (*These vectors form an orthogonal basis for* W: *check!*)

and let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find the point in W closest to $\boldsymbol{y}$ and find the distance from $\boldsymbol{y}$ to W.

$$\widehat{\boldsymbol{y}} = \frac{\boldsymbol{y}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{y}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \frac{\boldsymbol{y}\cdot\boldsymbol{u_3}}{\boldsymbol{u_3}\cdot\boldsymbol{u_3}}\boldsymbol{u_3}$$

$$= \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{6}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{0}{2}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \text{the point in W}$$

closest to $\boldsymbol{y}$.

$\begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$ is the "best approximation" to $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ using a point from W.

The distance from $\boldsymbol{y}$ to $W = ||\boldsymbol{z}|| = ||\,\boldsymbol{y} - \widehat{\boldsymbol{y}}\,|| = ||\begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}||$

$= \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

We can also think of $||\,\boldsymbol{y} - \widehat{\boldsymbol{y}}\,||$ as being the <u>error</u> committed when $\widehat{\boldsymbol{y}}$ is used as an approximation for $\boldsymbol{y}$. If $\boldsymbol{v} \in W$ and $\boldsymbol{v} \neq \widehat{\boldsymbol{y}}$, then $||\boldsymbol{y} - \boldsymbol{v}|| > \frac{\sqrt{3}}{3}$.

Example (continuation from first side)

An orthonormal basis for the same subspace W of $\mathbb{R}^4$ can be obtained by normalizing the original basis to get

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

Let $U = \begin{bmatrix} 0 & 2/\sqrt{6} & 0 \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{6} & -1/\sqrt{2} \\ 0 & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ and $U^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/\sqrt{6} & 0 & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Then $UU^T = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

UU^T is 4×4 and defines a mapping $T: \mathbb{R}^4 \to \mathbb{R}^4$, $T(\boldsymbol{y}) = UU^T\boldsymbol{y}$

UU^T is the matrix from the orthogonal projection mapping of $\mathbb{R}^4$ onto W :

$$UU^T\boldsymbol{y} = \widehat{\boldsymbol{y}}$$

For example, from the other side, $UU^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$.

The range ($=$ column space) for UU^T is the subspace W of $\mathbb{R}^4$.